Use the Heisenberg equation of motion for operators $O$

$$i\hbar \frac{d}{dt} O = i\hbar \frac{d}{dt} O_{\text{explicit}} + [O, H]$$  \hspace{1cm} (1)$$

where the Hamiltonian is that for the Dirac particle

$$H = c(\bar{\alpha} \cdot \bar{p} + mc\beta)$$  \hspace{1cm} (2)$$

and the “explicit” part of the time-derivative is necessary if the operator $O$ has an explicit dependence on $t$. Answer the following questions.

1. Show that the momentum $\bar{p} = -i\hbar(\partial / \partial \vec{x})$ is conserved.

2. Show that the orbital angular momentum $\vec{L} = \vec{x} \times \vec{p}$ is not conserved.

3. Show that the spin angular momentum

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma} = \frac{\hbar}{2} \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$  \hspace{1cm} (3)$$

is not conserved, but the total angular momentum $\vec{J} = \vec{L} + \vec{S}$ is conserved. (Notation changed from the class: $\Sigma$ is used for $4 \times 4$ matrices, while $\sigma$ always for Pauli matrices to avoid possible confusions.)

4. Show that the helicity $h = (\vec{S} \cdot \vec{p})/|\vec{p}|$ is conserved.

5. Show that the possible eigenvalues of the helicity are $\pm \hbar/2$.

6. Show that the above $\Sigma$ matrix can be written in a Lorentz-covariant way,

$$\Sigma^i = \frac{i}{2} \epsilon^{ijk} [\gamma^j, \gamma^k].$$  \hspace{1cm} (4)$$

7. Guess the Lorentz-covariant generalization $M^{\mu\nu}$ of the angular momentum $M^{ij} = \epsilon^{ijk} J^k$. Show that $M^{0i}$ are also conserved.