1. We would like to check the gauge invariance of the QED Lagrangian

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\bar{D} - m)\psi, \quad (1) \]

where \( D_\mu = \partial_\mu - ieQ A_\mu \). The gauge transformation is given by

\[ \psi'(x) = e^{iQ\omega(x)}\psi(x), \quad A'_\mu(x) = A_\mu(x) + \frac{1}{e} \partial_\mu \omega \quad (2) \]

(a) Show that \( D'_\mu \psi' = e^{iQ\omega} D_\mu \psi \).

(b) Show that \( [D_\mu, D_\nu] \psi = -ieQ F_{\mu\nu} \psi \). Note that this implies that \( [D'_\mu, D'_\nu] \psi' = -ieQ F'_{\mu\nu} \psi' = e^{iQ\omega}(-ieQ F_{\mu\nu} \psi) \) and hence \( F'_{\mu\nu} = F_{\mu\nu} \).

(c) Using the above results, show that \( \mathcal{L}(\bar{\psi}', \psi', A') = \mathcal{L}(\bar{\psi}, \psi, A) \).

2. We would like to check the gauge invariance of the Lagrangian of non-abelian gauge theories

\[ \mathcal{L} = -\frac{1}{2} TrF_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\bar{D} - m)\psi, \quad (3) \]

where \( D_\mu = \partial_\mu - ig A_\mu \) with the matrix form \( A_\mu = A_\mu^a T^a \). The gauge transformation is given by

\[ \psi'(x) = U(x)\psi(x), \quad A'_\mu(x) = U(x)A_\mu(x)U(x)^{-1} + \frac{i}{g} U\partial_\mu U^{-1}. \quad (4) \]

(a) Show that \( D'_\mu \psi' = UD_\mu \psi \).

(b) Show that \( D'_\mu = U D_\mu U^{-1} \).

(c) Define \( F_{\mu\nu} \) by \( [D_\mu, D_\nu] \psi = -ig F_{\mu\nu} \psi \). Show that \( [D'_\mu, D'_\nu] \psi' = -ig F'_{\mu\nu} \psi' = U(-ig F_{\mu\nu} \psi) \) and hence \( F'_{\mu\nu} = UF_{\mu\nu} U^{-1} \).

(d) Using the above results, show that \( \mathcal{L}(\bar{\psi}', \psi', A') = \mathcal{L}(\bar{\psi}, \psi, A) \).

(e) Show that \( F_{\mu\nu} = F_{\mu\nu}^a T^a \), \( (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c) T^a \).

(f) Under infinitesimal transformations \( U = e^{i\omega^a T^a} = 1 + i\omega^a T^a + O(\omega^2) \), show that \( A'_\mu = A_\mu + \frac{1}{g} D_\mu \omega + O(\omega^2) \), where \( \omega = \omega^a T^a \) and \( D_\mu \omega = \partial_\mu \omega - ig [A_\mu, \omega] \). Similarly, show that \( F'_{\mu\nu} = F_{\mu\nu} - i [F_{\mu\nu}, \omega] + O(\omega^2) \).