1. The $P$ and $C$ transformation of fermion pair bound states. It is customary to use Pauli–Dirac representation for $\gamma$-matrices when one is dealing with non-relativistic fermions, especially their bound states. You first start with the solutions at rest $p^\mu = \pm m(1, 0, 0, 0)$ which are eigenstates of $s_z$ (spin along the $z$ axis). Then you boost the Lorentz frame to obtain four-momentum along arbitrary directions. In this way, you obtain two positive energy solutions labeled as $u(p, s)$, where $s = \pm 1/2$ is the eigenvalue of $s_z$ before the boost. Similarly, you obtain two negative energy solutions $v(p, s)$. You can find explicit expressions for $u(p, s)$ and $v(p, s)$ in many textbooks, e.g. the one by Bjorken and Drell. By defining $u(p, s)$ and $v(p, s)$ in the above manner, you can choose your basis such that the following relations hold (far easier than the helicity basis!):

$$\gamma^0 u(\vec{p}; s) = u(-\vec{p}, s), \quad \gamma^0 v(\vec{p}; s) = -v(-\vec{p}, s),$$

(1)

$$i\gamma^0 \gamma^2 T u(\vec{p}; s) = v(\vec{p}, s), \quad i\gamma^0 \gamma^2 T v(\vec{p}; s) = u(\vec{p}, s).$$

(2)

We expand the Dirac field in the usual way also with this basis, i.e.,

$$\psi(x) = \int d\vec{p} \sum_s (a(p, s)u(p, s)e^{-ip \cdot x} + b^\dagger(p, s)v(p, s)e^{ip \cdot x})$$

(3)

and define the parity and charge conjugation by

$$P \psi(x, t)P = \gamma^0 \psi(-x, t)$$

(4)

$$C \psi(x, t)C = i\gamma^0 \gamma^2 T \bar{\psi}(x, t)$$

(5)

(1) Show that the mode operators satisfy the following relations, $Pa(\vec{p}, s)P = a(-\vec{p}, s), \; Pb(\vec{p}, s)P = -b(-\vec{p}, s)$.

(2) Show that the mode operators satisfy the following relations, $Ca(\vec{p}, s)C = b(\vec{p}, s), \;Cb(\vec{p}, s)C = a(\vec{p}, s)$.

(3) Define a state $|L, L_z; S, S_z\rangle$ with $L = l$ and $S = 0$ by

$$|l, m; 0, 0\rangle = \int d^3\vec{p} \left[a^\dagger(\vec{p}, +1/2)b^\dagger(-\vec{p}, -1/2) - a^\dagger(\vec{p}, -1/2)b^\dagger(-\vec{p}, +1/2)\right] \times Y^l_m(\hat{\vec{p}})f(|\vec{p}|)|0\rangle$$

(6)

Show that $P|l, m; 0, 0\rangle = -(-1)^l|l, m; 0, 0\rangle$. Here, $Y^l_m(\hat{\vec{p}})$ is defined by the direction of $\hat{\vec{p}} = \vec{p}/|\vec{p}|$ while $f(|\vec{p}|)$ is the “radial” part which depends only on the size of the momentum.
(4) Show that $C|l, m; 0, 0⟩ = (-1)^l|l, m; 0, 0⟩$. Since a photon has an odd eigenvalue under $C$, $l = 0$ state can decay into two photons. Examples include a para-positronium ($L = S = 0$ bound state of an electron and a positron), $\pi^0$ meson and $\eta^0$ meson (both $u\bar{u}$ and $d\bar{d}$ bound states in $L = S = 0$ channel).

(5) Define a state with $L = l$ and $S = 1$ by

$$|l, m; 1, 1⟩ = \int d^3\vec{p}[a_+^\dagger(\vec{p}, +1/2)b_+^\dagger(-\vec{p}, +1/2)]Y_0^l(\hat{\vec{p}})f(|\vec{p}|)|0⟩ \quad (7)$$

Show that $P|l, m; 1, 1⟩ = -(-1)^l|l, m; 1, 1⟩$. This is the same eigenvalue as the $S = 0$ case.

(6) Show that $C|l, m; 1, 1⟩ = -(-1)^l|l, m; 1, 1⟩$, which is the opposite eigenvalue from the $S = 0$ case. $l = 0$ state can hence decay into three photons. Examples include an ortho-positronium ($L = 0, S = 1$) decaying into three photons, and a $J/\psi$ particle ($L = 0, S = 1$ bound state of charm and anti-charm quark) which decays into three “gluons”.

**Note** It is usually summarized as $P = (-1)^{L+1}$ and $C = (-1)^{L+S}$.

2. **$CP$**. There are neutral Kaons, $K^0$ and $\bar{K}^0$, distinguished by their “strangeness”, 1 and $-1$, respectively. In the quark model, they are $d\bar{s}$ and $s\bar{d}$ bound states. Assume that $CP$ is a good quantum number.

(1) Both of them are $L = S = 0$ bound states. Are they scalars or pseudo-scalars?

(2) Assume $C|K^0⟩ = |\bar{K}^0⟩$ and vice versa. Write down eigenstates of $CP$ operator as linear combinations of $|K^0⟩$ and $|\bar{K}^0⟩$. The $CP = 1$ state is called $K_1$, and $CP = -1$ state $K_2$.

(3) Another neutral meson, $\pi^0$ is a bound state of $u\bar{u}$ and $d\bar{d}$ in $L = S = 0$ channel. (Never mind it is actually a $u\bar{u} - d\bar{d}$ combination.) If you have two $\pi^0$ with no relative angular momentum, show that it is a state with $CP = 1$, using a similar technique as in Problem 1., but with creation operators of a Klein-Gordon field. Therefore, $K_1$ can decay into $2\pi^0$ but $K_2$ cannot.

(4) Can $K_2$ decay into $3\pi^0$? (Assume it is kinematically allowed)

**Note** Experimentally, it was observed that $K_2$ can actually decay occasionally into $2\pi^0$. This was the only pheomenon known to 1998 which violates $CP$ invariance.