

**HW #4 (221b)**

1)

a) Set $h=m=a=1$.

\[
\delta_0[k, \gamma] := \frac{1}{2i} \log \left[ \frac{1 + \frac{2\gamma}{k} e^{-ik} \sin[k]}{1 + \frac{2\gamma}{k} e^{ik} \sin[k]} \right]
\]

\[
\sigma_0[k, \gamma] := \frac{4\pi}{k^2} \sin[\delta_0[k, \gamma]]^2
\]

($\gamma=100$): Large $\gamma$ emulates hard sphere scattering except at resonances. The graph below essentially matches that of the hard sphere cross section $\frac{4\pi}{k^2} \sin^2 ka$; in particular, $\sigma_0 \to 4\pi$ as $k \to 0$.

![Graph 1](image1.png)

When we zoom in ($\gamma=100$ still), we see the expected resonances at $ka \approx n\pi$, corresponding to the almost-bound states inside the shell.

![Graph 2](image2.png)

($\gamma=10$): For middling $\gamma$, the resonances are shifted towards smaller $k$ values according to equation (III.54) of the lecture notes. The peaks are higher and wider than the $\gamma=100$ resonances because of the increased width $\Gamma$. (See equation III.63, which is valid for poles close to the real axis.)
For small $\gamma$, we can’t distinguish the resonances from the oscillatory background. By III.52, small $\gamma$ means poles far from the real axis, so the disappearance of the resonances confirms that the particle doesn’t ‘feel’ poles far from its real momentum.

The wavefunction, defined piecewise on the intervals $r=(0,1)$, $r=(1,\infty)$, is

$$rR[r_-, t_-] := \begin{cases} \text{If}[r < 1, \text{Sin}[ (k0 - i \kappa) r] E^{-iE_0 t} E^{-i\Gamma t/2}, \text{Sin}[ (k0 - i \kappa) ] E^{i(k0 - i \kappa) (r-1)} E^{-iE_0 t} E^{-i\Gamma t/2} ] ; \end{cases}$$

Double click on the graph to watch the animation ($\gamma=5$):

$$\text{Do}[\text{Plot}[\text{Abs}[rR[r, i]]^2 / \{r \rightarrow 5, t \rightarrow 2 i\}, \{r, 0, 20\}, \text{PlotRange} \rightarrow \{0, 2\}], \{i, 0, 20\}]$$
We can show analytically that the decrease in probability inside the shell is equal to the probability flux flowing out through the shell. With \( \rho = |\psi|^2 \), \( j = \frac{-\hbar}{2mi} (\psi^* \nabla \psi - \psi \nabla \psi^*) \), we want to show \( \int \frac{\partial \rho}{\partial t} \, dV + \int j \cdot da = 0 \). In particular, we integrate \( \frac{\partial \rho}{\partial t} \) over the volume inside \( r < a \) and \( j \) over the sphere at \( r = a \). For the \( j \) integral, we can use the expression for the wavefunction just inside or just outside \( r = a \). The problem asks that we use the latter.

Using the above wavefunction with all constants restored,

\[
\frac{\partial \rho}{\partial t} = \frac{-\hbar}{\hbar} \rho = \frac{-\hbar}{2\hbar^2} \left( \cosh 2\kappa r - \cos 2k_0 r \right) e^{-\Gamma \hbar},
\]

so that

\[
\int \frac{\partial \rho}{\partial t} \, dV = \int_0^a \frac{\partial \rho}{\partial t} \, 4\pi r^2 = \frac{-2\pi \hbar}{\hbar} \left( \frac{\sinh 2\kappa a}{2\kappa} - \frac{\sin 2k_0 a}{2k_0} \right) e^{-\Gamma \hbar}.
\]

A similar computation for \( j > a \) wavefunction gives

\[
j = \frac{\hbar}{2mi} \left( i \frac{\hbar}{\hbar} + i k^* \right) \rho = \frac{\hbar k}{\hbar} \frac{r}{\hbar} \left( \cosh 2\kappa r - \cos 2k_0 r \right) e^{-\Gamma \hbar},
\]

and

\[
\int_{r=a} j \cdot da = 4\pi a^2 j = \frac{2\pi \hbar k_0}{m} \left( \cosh 2\kappa a - \cos 2k_0 a \right) e^{-\Gamma \hbar}.
\]

Plugging in \( \Gamma = \frac{2\hbar^2 k_0 \kappa}{m} \),

\[
\int \frac{\partial \rho}{\partial t} \, dV + \int j \cdot da = 0 \iff \frac{2\pi \hbar k}{m} \left( \frac{\sinh 2\kappa a}{2\kappa} - \frac{\sin 2k_0 a}{2k_0} \right) e^{-\Gamma \hbar} + \frac{2\pi \hbar k_0}{m} \left( \cosh 2\kappa a - \cos 2k_0 a \right) e^{-\Gamma \hbar} = 0 \iff \sinh 2\kappa a - \frac{\kappa}{k_0} \sin 2k_0 a = \cosh 2\kappa a - \cos 2k_0 a.
\]

To demonstrate this last equality, we must use the condition we developed when finding this solution to the Schrodinger equation, i.e. the condition from matching the discontinuity in derivatives at \( r = a \) with the delta function singularity. Equivalently, this is the condition for having a pole in the scattering amplitude: \( e^{\Theta k \cdot a} = 1 - 2i \frac{\hbar^2 k}{2\hbar m} \). Plugging in \( k = k_0 - i\kappa \), the real and imaginary parts of this equation read

\[
e^{2\kappa a} \cos 2k_0 a = 1 - \frac{\hbar^2 k}{my} \quad \text{and} \quad e^{2\kappa a} \sin 2k_0 a = -\frac{\hbar^2 k}{my}.
\]

Plugging these expressions into our condition
\[ \sinh 2 \kappa a - \frac{x}{k_0} \sin 2 k_0 a = \cosh 2 \kappa a - \cos 2 k_0 a \]

gives

\[ \sinh 2 \kappa a + \frac{\hbar^2 \kappa}{m y} e^{-2 \kappa a} = \cosh 2 \kappa a - (1 - \frac{\hbar^2 \kappa}{m y}) e^{-2 \kappa a} \]

or

\[ \sinh 2 \kappa a = \cosh 2 \kappa a - e^{-2 \kappa a}, \]

which does in fact hold.