HW #4 (221A), due Sep 22, 4pm

1. The uncertainty relation is actually there in classical waves, too.

(a) Show that a “localized light” of the form

$$E_y(x,t) = E_0 \sin 2\pi \nu \left( t - \frac{x}{c} \right) e^{-\left( \frac{x-ct}{2\sigma^2} \right)}$$

is a solution to the (one-dimensional) Maxwell equation. Sketch its shape. Calculate the “uncertainty” in the position $\Delta x$.

(b) Using Fourier analysis, determine the frequency of the light and its dispersion $\Delta \nu$. What is the product $\Delta x \Delta \nu$, and how small can it be?

Note However, the “uncertainty principle” in this case is purely classical, without involving $\hbar$. Only when you want to interpret the frequency $\nu$ as the momentum of the photon $p = \hbar \nu / c$, it becomes the quantum mechanical uncertainty principle.

2. The Gaussian wave packet represents a particle traveling in a “tight pack,”

$$\psi(x) = \langle x | \psi \rangle = Ne^{ipx/\hbar} e^{-\left( x-x_0 \right)^2/4d^2}.$$ 

Below, $X$ and $P$ are operators while $x$ and $p$ are numbers.

(a) Work out the normalization constant $N$.

(b) Show that $\langle X \rangle = x_0$.

(c) Calculate $\Delta X$.

(d) Work out the wave function in the momentum space, $\phi(p) = \langle p | \psi \rangle$.

(e) Show that $\langle P \rangle = p$.

(f) Calculate $\Delta P$ and show this wave function is a “minimum uncertainty state,” $\Delta X \Delta P = \hbar / 2$.

3. The operator $U(a) = e^{iapx/\hbar}$ is a translation operator in space (here we consider only one dimension). To see this, we need to prove an identity

$$e^A Be^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} [A, [A, \cdots [A, \underbrace{B}_{n}] \cdots]]$$

$$= B + [A, B] + \frac{1}{2!} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \cdots$$
(a) Consider $B(t) = e^{tA}Be^{-tA}$, where $t$ is a real parameter. Show \[ \frac{d}{dt}B(t) = e^{tA}[A, B]e^{-tA}. \]

(b) Obviously, $B(0) = B$ and therefore

\[ pB(1) = B + \int_0^1 dt \frac{d}{dt}B(t). \]

Now using the power series $B(t) = \sum_{n=0}^\infty t^n B_n$ and using the above integral expression, show $B_n = \frac{1}{n}[A, B_{n-1}]$.

(c) Show by induction that

\[ B_n = \frac{1}{n!} \left[ A, \underbrace{[A, [A, \cdots [A, B \cdots ]]}_n \right]. \]

(d) Use $B(1) = e^{A}Be^{-A}$ and prove the identity Eq. (3).

(e) Prove $e^{ipa/h}xe^{-ipa/h} = x+a$, showing $U(a)$ indeed translates space.