129B Solutions to HW #3

Problem 1.

Consider the Dirac equation with the vector potential A_μ:

$$[i\,\gamma^\mu\,(\partial_\mu - i\,e\,Q\,A_\mu) - m]\,\psi = 0$$

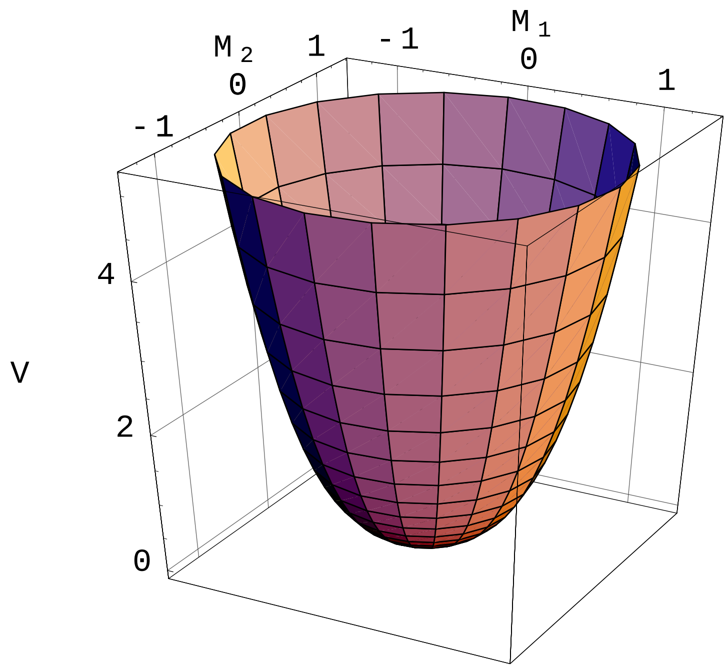
Under the gauge transformation

$$\begin{aligned}
&[i\,\gamma^\mu\,(\partial_\mu - i\,e\,Q\,A_\mu) - m]\,\psi \xrightarrow{\psi\to\psi',\,A\to A'} [i\,\gamma^\mu\,(\partial_\mu - i\,e\,Q\,(A_\mu + \partial_\mu\,\chi)) - m]\,e^{ieQ\chi}\,\psi = \\
&i\,\gamma^\mu\,\partial_\mu\,(e^{ieQ\chi}\,\psi) + [i\,\gamma^\mu\,(-i\,e\,Q\,(A_\mu + \partial_\mu\,\chi)) - m]\,e^{ieQ\chi}\,\psi = \\
&i\,\gamma^\mu\,e^{ieQ\chi}\,(i\,e\,Q\,\partial_\mu\,\chi)\,\psi + i\,\gamma^\mu\,e^{ieQ\chi}\,\partial_\mu\psi + [i\,\gamma^\mu\,(-i\,e\,Q\,(A_\mu + \partial_\mu\,\chi)) - m]\,e^{ieQ\chi}\,\psi = \\
&i\,\gamma^\mu\,e^{ieQ\chi}\,\partial_\mu\psi + [i\,\gamma^\mu\,(-i\,e\,Q\,A_\mu) - m]\,e^{ieQ\chi}\,\psi = e^{ieQ\chi}[i\,\gamma^\mu\,(\partial_\mu - i\,e\,Q\,A_\mu) - m]\,\psi = 0
\end{aligned}$$

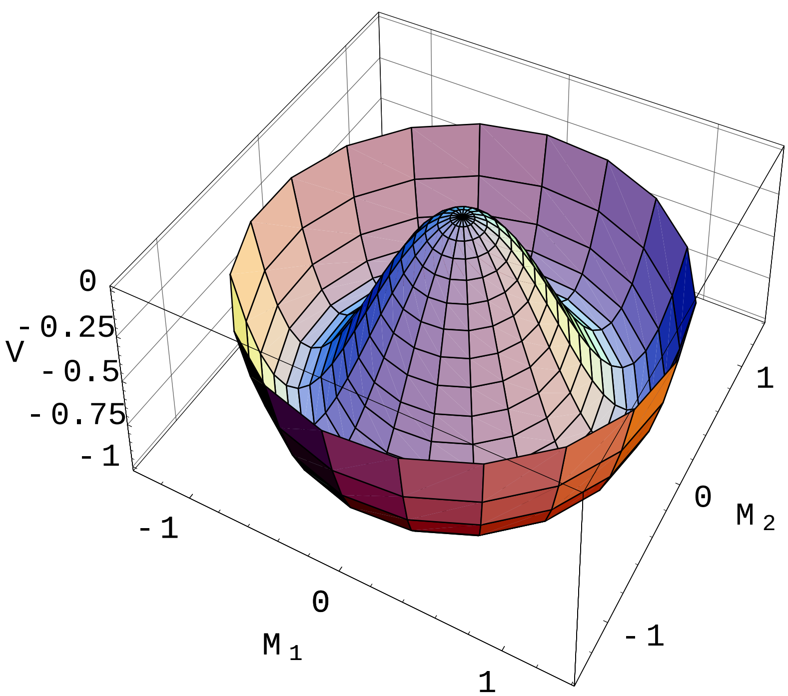
The original Dirac equation (with no A_μ field) is invariant under the *global* gauge transformation, i.e. the transformation $\psi \longrightarrow \psi' = e^{ieQ\chi}\,\psi$, where χ is independent of x^μ. If χ is different for different space-time points, an additional term arises when the derivative operator hits χ. To keep the equation invariant in this case, one introduces the A_μ field and postulates that it transforms in such a way as to exactly cancel the $\partial_\mu\,\chi$ term.

Problem 2.

Below are the plots showing the shape of the potential in two cases: $T > T_c$ and $T < T_c$. In the first case, the minimum is at the origin and the ground state does not break the symmetry. In the second case, however, any of the points on the circle $M_1^2 + M_2^2 = (T_c - T)/(2\,\lambda)$ can be a ground state. This means that below T_c the material becomes spontaneously magnetized, and the direction of the magnetization is random. Thus, a perfectly symmetric potential can yield a symmetry-breaking ground state.



$(T - T_c = 1,\ \lambda = 1)$



$(T - T_c = -2,\ \lambda = 1)$

Optional

Consider the Dirac equation for the left-handed electron - neutrino doublet:

$$\left[i\,\gamma^{\mu}\,(\partial_{\mu} - i\,g_2\,(T^{(a)})_{i\,j}\,W_{\mu}^{(a)}) - m\right]\psi_j = \left[i\,\gamma^{\mu}\left(\partial_{\mu} - i\,g_2\,\frac{(\sigma^{(a)})_{\text{ij}}}{2}\,W_{\mu}^{(a)}\right) - m\right]\psi_j =$$

$$\left[i\,\gamma^{\mu}\left(\partial_{\mu} - i\,g_2\,\frac{1}{2}\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}W_{\mu}^{1} + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}W_{\mu}^{2} + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}W_{\mu}^{3}\right)\right) - m\right]\begin{pmatrix}\nu\\ e\end{pmatrix} =$$

$$\left[i\,\gamma^{\mu}\left(\partial_{\mu} - i\,g_2\,\frac{1}{2}\begin{pmatrix}W_{\mu}^{3} & W_{\mu}^{1} - i\text{W}_{\mu}^{2}\\ W_{\mu}^{1} + i\text{W}_{\mu}^{2} & -W_{\mu}^{3}\end{pmatrix}\right) - m\right]\begin{pmatrix}\nu\\ e\end{pmatrix} =$$

$$\left[i\,\gamma^{\mu}\left(\partial_{\mu} - i\,g_2\,\frac{1}{2}\begin{pmatrix}W_{\mu}^{3} & \sqrt{2}\,W_{\mu}^{+}\\ \sqrt{2}\,W_{\mu}^{-} & -W_{\mu}^{3}\end{pmatrix}\right) - m\right]\begin{pmatrix}\nu\\ e\end{pmatrix}$$

The off-diagonal terms $\sqrt{2}\,W_{\mu}^{\pm}$ connect the electron and neutrino states. The diagonal terms $\pm W_{\mu}^{3}$ look like the A_{μ} vector in Problem 1, but implies the incorrect assign charge assignments of 1 and -1 to the neutrino and electron correspondingly and, hence, cannot be identified with the photon.

There can be other reasons why W_{μ}^{3} cannot be the photon. For example, it couples to $W_{\mu}^{\pm}$ through the non-Abelian interaction, and the strenth of that interaction is given by the weak coupling constant g_2, which is larger than the electric charge e (See HW #2). You can, however, try to save the situation by assuming that W_{μ}^{3} makes only part of the photon: $A_{\mu} = \sin\theta\, W_{\mu}^{3} + \cos\theta\, B_{\mu}$, and the other part B_{μ} does not couple to $W_{\mu}^{\pm}$. Then the coupling strength would be given by $g_2 \sin\theta$, which you can then try to identify with the elementary electric charge e. Remarkably, this is exactly how the Nature chooses to do it!