Brief Introduction to Tensor Calculus

I assume that you are familiar with the ordinary vector calculus (if not, let me know). The tensor calculus is basically the same as the vector calculus, but you can deal with many indices than just vectors.

For instance, an inner product of two vectors $\vec{a}$ and $\vec{b}$ are usually written as $\vec{a}\cdot\vec{b}$, but in terms of components, we write it as $a^i b^i$. Here, the sum over i is implicit.

An exterior product of two vectors $\vec{a}\times\vec{b}$ is written given by the components as

$$\begin{pmatrix} a_y b_z - a_z b_y \\ a_x b_y - a_y b_x \\ a_z b_x - a_x b_z \end{pmatrix}. \tag{1}$$

Another way of writing it is

$$(\vec{a}\times\vec{b})^i = \epsilon^{ijk} a^j b^k \tag{2}$$

where ϵ^{ijk} is a totally anti-symmetric three-rank tensor "Levi-Civita symbol." Don't be put off by the name! It has three indices i, j, k, and it changes its sign if you interchange two of the indices. Therefore, it vanishes if two of the indices are the same. Even though there are $3 \times 3 \times 3 = 27$ independent choices for i, j, k, there are only 3! choices which don't vanish,

$$\epsilon^{123} = \epsilon^{231} = \epsilon^{312} = -\epsilon^{321} = -\epsilon^{213} = -\epsilon^{132} = 1. \tag{3}$$

Let's check that the above expression for the exterior product is indeed correct.

$$(\vec{a}\times\vec{b})^1 = \epsilon^{1jk} a^j b^k = \epsilon^{123} a^2 b^3 + \epsilon^{132} a^3 b^2 = a^2 b^3 - a^3 b^2 \tag{4}$$

This agrees with the usual definition. This ϵ tensor is a useful way to deal with exterior products.

Now comes the Pauli matrices. Let me prove

$$\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k \tag{5}$$

First of all, the Kronecker's delta is defined by

$$\delta^{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \tag{6}$$

When $i = j$, the second term with the epsilon tensor vanishes. For instance, $\sigma^1\sigma^1 = \delta^{11} + i\epsilon^{11k}\sigma^k = \delta^{11} = 1$, which is right. The same is true for $\sigma^2\sigma^2$ or

$\sigma^3\sigma^3$. When $i \neq j$, the first term vanishes. For instance, $\sigma^1\sigma^2 = \delta^{12} + i\epsilon^{12k}\sigma^k = 0 + i\epsilon^{123}\sigma^3 = i\sigma^3$. This is also correct. One can also check all other combinations of i, j.

Another way of expressing the same relation is the commutator and anti-commutator. First the commutator:

$$[\sigma^i, \sigma^j] = \sigma^i\sigma^j - \sigma^j\sigma^i = (\delta^{ij} + i\epsilon^{ijk}\sigma^k) - (\delta^{jk} + i\epsilon^{jik}\sigma^k) = (\delta^{ij} - \delta^{ji}) + i(\epsilon^{ijk} - \epsilon^{jik})\sigma^k = 2i\epsilon^{ijk}\sigma^k. \tag{7}$$

This is the same as the commutation relation of the angular momentum operator $[J^i, J^j] = i\epsilon^{ijk}J^k$, if we identify $J^i = \sigma^i/2$. This is nothing but the Pauli's spin matrices. Secon the anti-commutator:

$$\{\sigma^i, \sigma^j\} = \sigma^i\sigma^j + \sigma^j\sigma^i = (\delta^{ij} + i\epsilon^{ijk}\sigma^k) + (\delta^{jk} + i\epsilon^{jik}\sigma^k) = (\delta^{ij} + \delta^{ji}) + i(\epsilon^{ijk} + \epsilon^{jik})\sigma^k = 2\delta^{ij}. \tag{8}$$

The following identity is useful:

$$\delta^{ij}b^j = b^i \tag{9}$$

This is easy to prove. For instance, $\delta^{1j}b^j = \delta^{11}b^1 + \delta^{12}b^2 + \delta^{13}b^3 = b^1$.

Finally, the product which we wanted to calculate:

$$\begin{aligned}
(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) &= (\sigma^i a^i)(\sigma^j b^j) \\
&= \sigma^i\sigma^j a^i b^j \\
&= (\delta^{ij} + i\epsilon^{ijk}\sigma^k)a^i b^j \\
&= \delta^{ij}a^i b^j + i\epsilon^{ijk}a^i b^j \sigma^k \\
&= a^i b^i + i(\epsilon^{kij}a^i b^j)\sigma^k \\
&= \vec{a} \cdot \vec{b} + i(\vec{a} \times \vec{b}) \cdot \vec{\sigma}.
\end{aligned} \tag{10}$$

Using this formula, we now obtain:

$$\begin{aligned}
&\left(\vec{\sigma} \cdot \left(\vec{p} - \frac{e}{c}\vec{A}\right)\right)^2 = \left(p^i - \frac{e}{c}A^i\right)\left(p^i - \frac{e}{c}A^i\right) + i\epsilon^{kij}\left(p^i - \frac{e}{c}A^i\right)\left(p^j - \frac{e}{c}A^j\right)\sigma^k \\
&= \left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + i\epsilon^{kij}\frac{1}{2}\left[\left(p^i - \frac{e}{c}A^i\right)\left(p^j - \frac{e}{c}A^j\right) - \left(p^j - \frac{e}{c}A^j\right)\left(p^i - \frac{e}{c}A^i\right)\right]\sigma^k \\
&= \left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + i\epsilon^{kij}\frac{1}{2}\left[p^i - \frac{e}{c}A^i, p^j - \frac{e}{c}A^j\right]\sigma^k \\
&= \left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + i\epsilon^{kij}\frac{1}{2}\left(-i\hbar\nabla^i\frac{-e}{c}A^j + i\hbar\nabla^j\frac{-e}{c}A^i\right)\sigma^k
\end{aligned}$$

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\begin{aligned}
&= \left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + i\epsilon^{kij}\frac{1}{2}\left(-i\hbar\frac{-e}{c}\epsilon^{ijl}B^l\right)\sigma^k \\
&= \left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + \hbar\frac{-e}{c}B^k\sigma^k \\
&= \left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + \hbar\frac{-e}{c}\vec{\sigma}\cdot\vec{B} \qquad (11)
\end{aligned}
$$

In the last step, I used the identity: $\epsilon^{ijk}\epsilon^{ijl} = 2\delta^{kl}$. This can be easily checked. Take $k = l = 3$. Then, $\epsilon^{ij3}\epsilon^{ij3} = \epsilon^{123}\epsilon^{123} + \epsilon^{213}\epsilon^{213} = 2 = 2\delta^{33}$.