Feynman Rules for QED

The Feynman rules:

1. Initial state electron (or particle in general): $u(p)$.
2. Final state electron (or particle in general): $\bar{u}(p)$.
3. Initial state positron (or anti-particle in general): $\bar{v}(p)$.
4. Final state positron (or anti-particle in general): $v(p)$.
5. Photon propagator: $\dfrac{-ig^{\mu\nu}}{q^2}$.
6. Electron propagator: $\dfrac{i}{\not{p} - m}$, where $\not{p} = \gamma^\mu p_\mu$.
7. Electron-photon vertex: $-ieQ\gamma^\mu$ with $Q = -1$. For general particles, change Q appropriately.
8. Conserve four-momenta at every vertices.

$$e^-e^+ \to \mu^-\mu^+$$

The Feynman amplitude for $e^-(k), e^+(\bar{k}) \to \mu^-(p)\mu^+(\bar{p})$. k, $\bar{k}$, p, $\bar{p}$ denote the four-momenta of initial and final state particles. The process goes through the s-channel photon exchange, *i.e..*, e^-e^+ annihilate into a photon by a vertex $ie\gamma^\mu$, then the photon "progagates" with the propagator $-ig_{\mu\nu}/q^2$ with $q^\mu = (k+\bar{k})^\mu = (p+\bar{p})^\mu$, and the photon converts to $\mu^-\mu^+$ by another vertex $ie\gamma^\nu$. Therefore, the amplitude is given by

$$i\mathcal{M} = \bar{u}(p)(ie\gamma^\nu)v(\bar{p})\frac{-ig_{\mu\nu}}{q^2}\bar{v}(\bar{k})(ie\gamma^\mu)u(k). \tag{1}$$

We first simplify it to

$$\mathcal{M} = \frac{e^2}{s}\bar{u}(p)\gamma_\mu v(\bar{p})\bar{v}(\bar{k})\gamma^\mu u(k), \tag{2}$$

where $s = q^2$ is the squared center-of-momentum energy.

Now we calculate the amplitude explicitly. We fix the reference frame to the center-of-momentum frame of the collision, with four-momenta

$$k^\mu = E(1,0,0,1), \tag{3}$$
$$\bar{k}^\mu = E(1,0,0,-1), \tag{4}$$
$$p^\mu = E(1,\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta), \tag{5}$$
$$\bar{p}^\mu = E(1,-\sin\theta\cos\phi,-\sin\theta\sin\phi,-\cos\theta), \tag{6}$$

with $E = \sqrt{s}/2$. Here and below, we neglect the masses completely which is valid if $E \gg m$.

Let us first consider the case of $e^-_R e^+_L \to \mu^-_R\mu^+_L$, where the subscript R refers to the helicity $+1/2$ state and L to $-1/2$ state. The initial state electron is described by the wave function $u_+(k)$, and since the four-momentum k^μ is given by $\theta = 0$, $\phi = 0$, we find

$$u_+(k) = \frac{1}{\sqrt{E}}\begin{pmatrix} E\chi_+(k) \\ k\chi_+(k) \end{pmatrix} = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{7}$$

Here and below, we use the relativistic formula $E = \sqrt{k^2+m^2} = k$ for massless particles. The initial state positron is described by the wave function $\bar{v}_-(\bar{k})$. Since the four-momentum $\bar{k}$ is given by $\theta = \pi$, $\phi = \pi$ (the choice of ϕ is arbitrary if

$\theta = 0$ or π; the ambiguity in ϕ results in an ambiguity in the overall phase of the amplitude, which is unphysical). The wave function is given by

$$v_-(\bar{k}) = \frac{1}{\sqrt{E}} \begin{pmatrix} k\chi_+(\bar{k}) \\ E\chi_+(\bar{k}) \end{pmatrix} = \sqrt{E} \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \end{pmatrix}. \tag{8}$$

Then the combination $\bar{v}(\bar{k})\gamma^\mu u(k)$ can be calculated just by a matrix algebra. Recall that $\bar{v} = v^\dagger \gamma^0$, and $\gamma^0\gamma^0 = 1$, $\gamma^0\gamma^i = \alpha^i$ for $i = 1, 2, 3$. Therefore,

$$\bar{v}_-(\bar{k})\gamma^0 u_+(k) = v_-(\bar{k})^\dagger u_+(k) = 0, \tag{9}$$

$$\bar{v}_-(\bar{k})\gamma^i u_+(k) = v_-(\bar{k})^\dagger \alpha^i u_+(k) = 2E(0, -1)\sigma^i \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{10}$$

and we find

$$\bar{v}_-(\bar{k})\gamma^\mu u_+(k) = 2E(0, -1, -i, 0). \tag{11}$$

For the final state particles, we use the wave function

$$u_+(p) = \frac{1}{\sqrt{E}} \begin{pmatrix} E\chi_+(p) \\ p\chi_+(p) \end{pmatrix} = \sqrt{E} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\phi} \\ \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\phi} \end{pmatrix}, \tag{12}$$

and for $\bar{p}$, we substitute $\theta \to \pi - \theta$, $\phi \to \phi + \pi$,

$$v_-(\bar{p}) = \frac{1}{\sqrt{E}} \begin{pmatrix} \bar{p}\chi_+(\bar{p}) \\ E\chi_+(\bar{p}) \end{pmatrix} = \sqrt{E} \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}\, e^{i\phi} \\ \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}\, e^{i\phi} \end{pmatrix}. \tag{13}$$

Now we can calculate the combination $\bar{u}(p)\gamma^\nu v(\bar{p})$ necessary for the amplitude.

$$\bar{u}_+(p)\gamma^0 v_-(\bar{p}) = u_+(p)^\dagger v_-(\bar{p}) = 0, \tag{14}$$

$$\bar{u}_+(p)\gamma^i v_-(\bar{p}) = u_+(p)^\dagger \alpha^i v_-(\bar{p}) = 2E(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\, e^{-i\phi})\sigma^i \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}\, e^{i\phi} \end{pmatrix}, \tag{15}$$

and we find

$$\bar{u}_+(p)\gamma^\nu v_-(\bar{p}) = 2E(0, -\cos^2\frac{\theta}{2}e^{i\phi} + \sin^2\frac{\theta}{2}e^{-i\phi}, i\cos^2\frac{\theta}{2}e^{i\phi} + i\sin^2\frac{\theta}{2}e^{-i\phi}, 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}). \tag{16}$$

Putting pieces together, we find the amplitude (2):

$$\begin{aligned} \mathcal{M} &= \frac{e^2}{s}\bar{u}_+(p)\gamma_\mu v_-(\bar{p})\bar{v}_-(\bar{k})\gamma^\mu u_+(k) \\ &= \frac{e^2}{s}4E^2(-2\cos^2\frac{\theta}{2}e^{i\phi}) \\ &= -e^2(1+\cos\theta)e^{i\phi}. \end{aligned} \tag{17}$$

Note that the contraction of the Lorentz index μ requires a negative sign for all spatial components. The θ dependence of the amplitude makes a good sense. Because we started with e_R^- and e_L^+ along the z-axis, the initial state has a total spin of $+1$ along the positive z-axis. On the other hand, the final state $\mu_R^-\mu_L^+$ has also a total spin of $+1$ along the (θ, ϕ) direction. The angular momentum conservation forbids $\theta = \pi$ which makes the final state spin point to the negative z-axis, where the amplitude indeed vanishes. The amplitude is the largest when the final state spin points to the same direction as the initial state, $\theta = 0$. The ϕ dependence of the amplitude is only in its phase. Since the cross section is proportional to the amplitude absolute squared, there remains no ϕ dependence. This is expected because the collision of particles along the z axis is axially symmetric. In any case, this completes the calculation of the amplitude for this particular helicity combination.

For the helicity combination $e_L^-e_R^+ \to \mu_R^-\mu_L^+$, the only changes are in the initial state wave functions. We now have

$$u_-(k) = \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \qquad v_+(\bar{k}) = \sqrt{E}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{18}$$

This gives

$$\bar{v}_+(\bar{k})\gamma^\mu u_-(k) = 2E(0, 1, -i, 0). \tag{19}$$

We can reuse Eq. (16) for the final state part of the amplitude. We find the amplitude to be

$$\begin{aligned} \mathcal{M} &= \frac{e^2}{s}\bar{u}_+(p)\gamma_\mu v_-(\bar{p})\bar{v}_+(\bar{k})\gamma^\mu u_-(k) \\ &= \frac{e^2}{s}4E^2(-2\sin^2\frac{\theta}{2}e^{-i\phi}) \\ &= -e^2(1-\cos\theta)e^{i\phi}. \end{aligned} \tag{20}$$

Again the angular dependence can be intuitively understood in terms of angular momentum conservation.

For the helicity combinations $\mu_L^-\mu_R^+$, we find

$$\bar{u}_-(p)\gamma^\nu v_+(\bar{p}) = 2E(0, \cos^2\frac{\theta}{2}e^{-i\phi} - \sin^2\frac{\theta}{2}e^{i\phi}, i\cos^2\frac{\theta}{2}e^{-i\phi} + i\sin^2\frac{\theta}{2}e^{i\phi}, -2\cos\frac{\theta}{2}\sin\frac{\theta}{2}). \tag{21}$$

Then for $e_R^-e_L^+ \to \mu_L^-\mu_R^+$, the amplitude is

$$\mathcal{M} = -\frac{e^2}{s}(1-\cos\theta)e^{i\phi}, \tag{22}$$

and for $e_L^-e_R^+ \to \mu_L^-\mu_R^+$, the amplitude is

$$\mathcal{M} = -\frac{e^2}{s}(1+\cos\theta)e^{-i\phi}. \tag{23}$$

Other helicity combinations such as $e_R^-e_R^+$, $e_L^-e_L^+$, $\mu_R^-\mu_R^+$, $\mu_L^-\mu_L^+$ give vanishing amplitudes. This can be easily checked, and is a consequence of the "chirality conservation" which is true in the massless limit.

The four amplitudes, (17), (20), (22), and (23), are the only non-vanishing ones. In order to obtain the cross section, we use the general formula

$$d\sigma(i \to f) = \frac{1}{2s\bar{\beta}_i}|\mathcal{M}(i \to f)|^2 d\Phi_n. \tag{24}$$

In our case of massless initial state particles, $\bar{\beta}_i = 0$, and we need only the two-body phase space,

$$d\Phi_2 = \frac{\bar{\beta}_f}{8\pi}\frac{d\cos\theta}{2}\frac{d\phi}{2\pi}. \tag{25}$$

We have $\bar{\beta}_f = 1$ for the massless muons. For typical ring e^+e^- colliders, both electron and positron beams are not polarized. Therefore, we need to average the helicities $+1/2$ and $-1/2$ with 50:50 ratio. This average is done both for electron and positron. On the other hand, we are interested in the total production cross section of $\mu^+\mu^-$ and we just sum their helicities over. Then we find

$$\begin{aligned} d\sigma(e^-e^+ \to \mu^-\mu^+) = \frac{1}{4}\frac{1}{2s}\Big[&|\mathcal{M}(e_R^-e_L^+ \to \mu_R^-\mu_L^+)|^2 + |\mathcal{M}(e_L^-e_R^+ \to \mu_R^-\mu_L^+)|^2 \\ &+|\mathcal{M}(e_R^-e_L^+ \to \mu_L^-\mu_R^+)|^2 + |\mathcal{M}(e_L^-e_R^+ \to \mu_L^-\mu_R^+)|^2\Big]\frac{1}{8\pi}\frac{d\cos\theta}{2}\frac{d\phi}{2\pi}. \end{aligned} \tag{26}$$

The prefactor $1/4$ is from the helicity average. The sum of squared amplitudes is $4e^4(1+\cos^2\theta)$, and the ϕ integral is trivial, $\int d\phi/2\pi = 1$. The total cross section is obtained upon $\cos\theta$ integral,

$$\sigma(e^-e^+ \to \mu^-\mu^+) = \frac{1}{4}\frac{1}{2s}\frac{1}{8\pi}\int_{-1}^{1} 4e^4(1+\cos^2\theta)\frac{d\cos\theta}{2} = \frac{e^4}{16\pi s}\frac{4}{3}. \tag{27}$$

It is conventional to rewrite e in terms of the fine structure constant $\alpha = e^2/4\pi$, and we find

$$\sigma(e^- e^+ \to \mu^- \mu^+) = \frac{4}{3} \frac{\pi \alpha^2}{s}. \tag{28}$$

For production of Dirac particles $f\bar{f}$ other than muons, we change the vertex $ie\gamma^\nu$ for the muon to $-ieQ_f\gamma^\nu$ with charge Q_f of the particle, and we multiply the cross section by the appropriate multiplicities, *e.g.*, number of colors N_c. The general formula then reads as

$$\sigma(e^- e^+ \to f\bar{f}) = \frac{4}{3} \frac{\pi \alpha^2}{s} Q_f^2 N_c. \tag{29}$$

If the final state particle has spin 0 rather than 1/2 (Dirac), we instead have

$$\sigma(e^- e^+ \to f\bar{f}) = \frac{1}{3} \frac{\pi \alpha^2}{s} Q_f^2 N_c. \tag{30}$$

The angular distribution is also different, $\sin^2\theta$ rather than $1 + \cos^2\theta$, which can be understood in terms of angular momentum conservation again.