Midterm Solution Set

1. (a) From the Kamiokande paper, it is actually not completely clear if the angles they quote are measured from forward or backward. Despite what I've said to some of you, I feel that they are probably measured from forward, *i.e.*, $\theta = 0$ corresponds to a positron which goes along the same direction as the original neutrino. But I think both are equally valid interpretation of the paper, and accept answers based on either interpretation.

First of all, one can neglect the mass of the neutrino in the calculation of kinematics for the $\bar{\nu}_e p \to e^+ n$ scattering. Judging from the energies listed in the table, we can even ignore the mass of the electron. Then the four-momenta of $\bar{\nu}_e$, p, e^+, n can be written as

$$\begin{aligned} p_\nu &= \frac{E_\nu}{c}(1,0,0,1) && (1)\\ p_p &= m_p c(1,0,0,0) && (2)\\ p_e &= \frac{E_e}{c}(1,\sin\theta,0,\cos\theta) && (3)\\ p_n &= \frac{1}{c}(E_\nu + m_p c^2 - E_e, -E_e\sin\theta, 0, E_\nu - E_e\cos\theta). && (4)\end{aligned}$$

Since

$$\begin{aligned} p_n^2 c^2 = m_n^2 c^4 &= (E_\nu + m_p c^2 - E_e)^2 - (E_e \sin\theta)^2 - (E_\nu - E_e\cos\theta)^2 \\ &= m_p^2 c^4 + 2E_e m_p c^2 + 2E_\nu(m_p c^2 - E_e(1-\cos\theta)), \qquad (5)\end{aligned}$$

we can solve for E_ν:

$$E_\nu = \frac{m_n^2 c^4 - m_p^2 c^4 + 2E_e m_p c^2}{2m_p c^2 - 2E_e(1-\cos\theta)}\,. \qquad (6)$$

If the angle is defined from backward, $1-\cos\theta$ is replaced by $1+\cos\theta$. Using this formula, we can calculate the neutrino energies for all 12 events observed by Kamiokande.

event	t(sec)	E_e(MeV)	θ(deg)	E_ν(MeV)	E_ν(MeV)
1	0	20.0	18	22.2	21.3
2	0.107	13.5	15	15.2	14.8
3	0.303	7.5	108	8.84	8.89
4	0.324	9.2	70	10.6	10.6
5	0.507	12.8	135	14.2	14.4
6	0.686	6.3	68	7.66	7.63
7	1.541	35.4	32	39.4	36.9
8	1.728	21.0	30	23.3	22.4
9	1.915	19.8	38	21.9	21.2
10	9.219	8.6	122	9.94	10.0
11	10.433	13.0	49	14.6	14.4
12	12.439	8.9	91	10.3	10.3

Here, the first E_ν column corresponds to θ measured from backward, and the second from forward.

(b) Using only the first 9 events, we require that the arrival times for the energy range between 7.63–36.9 MeV (events 6 and 7) should not differ more than 2 seconds. For a small neutrino mass, the velocity can be written as

$$v = c\sqrt{1 - \frac{m^2c^4}{E^4}} \simeq c\left(1 - \frac{m^2c^4}{2E^4}\right), \tag{7}$$

and the time to reach the earth as

$$t \simeq \frac{L}{c}\left(1 + \frac{m^2c^4}{2E^4}\right). \tag{8}$$

Here, $L = 50$ kpc is the estimated distance between the Earth and SN1987A, and hence t is as long as 150 thousand years. After this long journey from SN1987A, the neutrinos with different energies arrived at the Earth within 2 seconds, which implies that the neutrinos must be very close to being massless. The difference between the arrival times for a neutrino with $E_{min} = 7.63$ MeV and $E_{max} = 36.9$ MeV is given by

$$\Delta t \simeq \frac{L}{2c} m^2c^4 \left(\frac{1}{E^4_{min}} - \frac{1}{E^4_{max}}\right). \tag{9}$$

By requiring $\Delta t < 2$ sec, we find

$$m < 6.6 \text{ eV}/c^2 \tag{10}$$

N.B. This analysis neglects error bars in angle and energy measurements, and also does not utilize the theoretical calculation of the burst profile and hence is rather crude. For a more detailed discussion, see, *e.g.*, W. D. Arnett and J. L. Rosner, *Phys. Rev. Lett.* **58**, 1906 (1987). Their result does not differ much from our crude analysis. You may have used a slightly different requirement from mine; it is OK as long as the order of magnitudes of the upper bound are the same.

2. (a) $p \to e^+\pi^0$, baryon number or electron number. (b) $\mu^- \to e^-e^-e^+$, muon number of electron number. (c) $n \to p\nu_e\bar{\nu}_e$, electric charge. (d) $\tau^- \to \mu\gamma$, tau number or muon number. (e) $n \to p\mu^-\bar{\nu}_\mu$, energy. (f) $K^0 \to \mu^+e^-$, muon number or electron number. (g) $\mu^- \to \pi^-\nu_\mu$, energy.

3. (a) $\rho \to \pi\pi$, $\omega \to \pi^+\pi^-\pi^0$. The charges of $\pi\pi$ in ρ decay are not specified in the booklet. The booklet assumes that you can figure out the rest of this problem.

(b) Since ρ has spin 1 and π's don't, the relative orbital angular momentum of $\pi\pi$ system must be $L = 1$, *i.e.*, P-wave. The angular wavefunction of the $\pi\pi$ system therefore is given by the spherical harmonics Y^1_m with $m = -1$, 0, or 1. The important feature of Y^l_m is that they are even functions for l even and odd for l odd. Therefore, $L = 1$ case gives an odd function for the relative coordinate, and hence the wave function changes its sign under the interchange of two πs. On the other hand, πs are bosons and hence their wave functions should not change their signs under the interchange of two pions. Therefore, the two-pion wave function must be $|\pi^+\pi^-\rangle - |\pi^-\pi^+\rangle$ (anti-symmetric) to compensate the sign from Y^1_m. However, this construction gives an identically vanishing wavefunction for $\pi^0\pi^0$ state, and hence $\rho \to \pi^0\pi^0$ cannot occur.

(c) By adding two $I = 1$, the possibilities for the total isospin are $I = 0$, 1, or 2. Since $I = 1$ has three states, there are $3 \times 3 = 9$ states in total, in agreement with the sum of number of states 1 ($I = 0$), 3 ($I = 1$) and 5 ($I = 2$). The number of symmetric states can be counted as $(3 \cdot 4)/2! = 6$, which coincides with the number of states for $I = 0, 2$, and the number of anti-symmetric states ${}_3C_2 = (3 \cdot 2)/2! = 3$ with the number of states for $I = 1$. Therefore, $I = 0, 2$ have symmetric wavefunctions and $I = 1$ anti-symmetric wavefunctions for the isospin part. The angular wavefunction must hence be symmetric for $I = 0, 2$ and anti-symmetric for $I = 1$ to make the total wavefunction symmetric under the intechange of two πs as required by Bose statistics. The relative orbital angular momentum must therefore be even for $I = 0$ and odd for $I = 1$.

(d) ω meson has spin 1 and $I = 0$. If ω were to decay into $\pi\pi$ state, the conservation of isospin requires that $\pi\pi$ state has $I = 0$ as well. Then the result from the previous problem implies that L has to be even. On the other hand, the conservation of angular momentum requires $L = 1$ and contradicts with the other requirement. This is why ω cannot decay into $\pi\pi$ state.

(e) The $\omega \to \pi\pi$ must be caused by an effect which does not respect isospin. One candidate is the electromagnetic interaction. In fact, $\omega = (u\bar{u} + d\bar{d})/\sqrt{2}$, and a photon can be emitted from quarks or anti-quarks to create a pair of quarks and they can combine into two π's. (Another possibility is that the small mass difference between u and d quarks violates isospin and results in a small mixing between ρ^0 and ω^0. Then ω contains a small $I = 1$ component and it can decay into $\pi\pi$.)

N.B. The isospin wavefunctions for $I = 1$ are indeed anti-symmetric, and can be written as

$$|1,1\rangle = \frac{1}{\sqrt{2}}\left(|\pi^+\pi^0\rangle - |\pi^0\pi^+\rangle\right), \tag{11}$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|\pi^+\pi^-\rangle - |\pi^-\pi^+\rangle\right), \tag{12}$$

$$|1,-1\rangle = \frac{1}{\sqrt{2}}\left(|\pi^0\pi^-\rangle - |\pi^-\pi^0\rangle\right). \tag{13}$$

Note that there is no $|\pi^0\pi^0\rangle$ consistent with the discussion in problem (b).

4. (a) Out of six possible states, totally symmetric combination of three quarks can be counted as $(6 \cdot 7 \cdot 8)/3! = 56$.

(b) The octet baryons have spin 1/2, and the total number of states is $8 \cdot 2 = 16$. The decouplet baryons have spin 3/2, and the total number of states is $10 \cdot 4 = 40$. The grant total is 56, and hence the octet and decouplet exhaust all possible states constructed from three quarks (in S-wave, as implicitly assumed).

(c) By acting J^-, we find

$$\sqrt{3}|\Delta^{++}, 1/2\rangle = |u^\uparrow u^\uparrow u^\downarrow\rangle + |u^\uparrow u^\downarrow u^\uparrow\rangle + |u^\downarrow u^\uparrow u^\uparrow\rangle. \tag{14}$$

The $|\Delta^{++}, 1/2\rangle$ state is given by $1/\sqrt{3}$ of the r.h.s above.

(d) By acting I^- on the above equation, we find

$$\begin{aligned} 3|\Delta^{+}, 1/2\rangle &= |u^\uparrow u^\uparrow d^\downarrow\rangle + |u^\uparrow u^\downarrow d^\uparrow\rangle + |u^\downarrow u^\uparrow d^\uparrow\rangle \\ &+ |u^\uparrow d^\uparrow u^\downarrow\rangle + |u^\uparrow d^\downarrow u^\uparrow\rangle + |u^\downarrow d^\uparrow u^\uparrow\rangle \\ &+ |d^\uparrow u^\uparrow u^\downarrow\rangle + |d^\uparrow u^\downarrow u^\uparrow\rangle + |d^\downarrow u^\uparrow u^\uparrow\rangle. \end{aligned} \tag{15}$$

The $|\Delta^{+}, 1/2\rangle$ state is given by 1/3 of the r.h.s above.

(e) The inner product of $|\Delta^{+}, 1/2\rangle$ and $|p, 1/2\rangle$ states is given by

$$\langle p, 1/2|\Delta^{+}, 1/2\rangle = \frac{1}{3}\frac{1}{3\sqrt{2}}(2+2+2-1-1-1-1-1-1) = 0. \tag{16}$$

Since eigenstates of Hamiltonian with different eigenvalues must be orthogonal to each other, this result is necessary. The check of total symmetry can be done as follows. There are $3! = 6$ possible permutations of the three quarks. The form given in the problem is already manifestly symmetric under the (three) cyclic permutations. The other three permutations are interchange of (1,2), (2,3) and (3,1) quarks. It is easy to check that the wave function does not change under these permutations. Therfore, the wavefunction given here is totally symmetric. Together with the totally anti-symmetric color wavefunction, the quarks have totally anti-symmetric wavefunction as required by Fermi statistics.

(f) The eigenvalue of I^3 operator is given by the sum of I^3 of individual quarks, $(1/2)+(1/2)+(-1/2) = 1/2$. Acting I^+ "raises" the only down quark to the up quark,

$$\begin{aligned} I^+|p, 1/2\rangle &= \frac{1}{3\sqrt{2}}\Big(2|u^\uparrow u^\uparrow u^\downarrow\rangle + 2|u^\uparrow u^\downarrow u^\uparrow\rangle + 2|u^\downarrow u^\uparrow u^\uparrow\rangle - |u^\uparrow u^\downarrow u^\uparrow\rangle \\ &\quad - |u^\downarrow u^\uparrow u^\uparrow\rangle - |u^\uparrow u^\downarrow u^\uparrow\rangle - |u^\uparrow u^\uparrow u^\downarrow\rangle - |u^\uparrow u^\uparrow u^\downarrow\rangle - |u^\downarrow u^\uparrow u^\uparrow\rangle\Big) \\ &= 0. \end{aligned} \tag{17}$$

(g) The check is similar to the previous problem.

(h) According to the assumption, the magnetic moment operator is given by

$$\vec{\mu} = \sum_i 2\frac{e_i}{2m_q}\vec{s}_i, \tag{18}$$

where $e_i = (2/3)e$ for up quarks and $-(1/3)e$ for down quarks. $\vec{s}_i$ are spin operators for each quark. The expectation value does not vanish only for μ_z, and is given by

$$\begin{aligned}\langle p, 1/2|\mu_z|p, 1/2\rangle &= 2\left(\frac{1}{3\sqrt{2}}\right)^2 \frac{e\hbar}{2m_q}\left[2^2\left(\frac{2}{3}\frac{1}{2}+\frac{2}{3}\frac{1}{2}-\frac{1}{3}\left(-\frac{1}{2}\right)\right)\times 3\right. \\ &\quad \left. +\left(\frac{2}{3}\frac{1}{2}+\frac{2}{3}\left(-\frac{1}{2}\right)-\frac{1}{3}\frac{1}{2}\right)\times 6\right] \\ &= \frac{e\hbar}{2m_q} \\ &= \frac{e\hbar}{2m_p}3. \end{aligned} \tag{19}$$

The booklet quotes $\mu = 2.8\mu_N$ where μ_N is the nuclear magneton $e\hbar/2m_p$ (see 1. PHYSICAL CONSTANTS). Given the crudeness of the approximation (*i.e.*, the assumption of free quarks despite strong gluon interactions), this agreement is remarkable.

N.B. One can repeat the same analysis for the neutron. The neutron wavefunction can be obtained from that of the proton by interchanging up quarks and down quarks. Therefore the magnetic moment is obtained by interchaning 2/3 and $-1/3$, and it is easy to find $\frac{e\hbar}{2m_n}(-2)$. This again agrees remarkably well with the value in the booklet: $\mu = -1.9\mu_N$.