Cross section, Decay Rate, Phase Space, Amplitude

Below, the natural unit $\hbar = c = 1$ is used throughout.
Lorentz-invariant phase space:

$$\int d\Phi_n = \int \prod_i^n \left(\frac{d^3\vec{p}_i}{(2\pi)^3 2E_i} \right) (2\pi)^4 \delta^4 (\sum_i^n p_i - P). \tag{1}$$

Here, $E_i = \sqrt{\vec{p}_i^2 + m_i^2}$ is the energy of the particle i of mass m_i. Each particle has four-momentum $p_i^\mu = (E_i, \vec{p}_i)$. The total four-momentum of the n-body system is P^μ.
Feynman Amplitude from the initial state i at $t = -\infty$ to the final state f at $t = \infty$:

$$\mathcal{M}(i \to f)(2\pi)^4 \delta^4 (\sum_i^n p_i - P) = \langle f, t = +\infty | i, t = -\infty \rangle = \lim_{T \to \infty} \langle f | e^{-iHT} | i \rangle. \tag{2}$$

The convention is determined by the normalization of single particle states:

$$\langle \vec{p}_1 | \vec{p}_2 \rangle = (2\pi)^3 2E \delta^3 (\vec{p}_1 - \vec{p}_2). \tag{3}$$

This normalization has an advantage of being Lorentz-invariant.
Differential Partial Decay Rate of a particle of mass M to the n-body final state f:

$$d\Gamma(i \to f) = \frac{1}{2M} |\mathcal{M}(i \to f)|^2 d\Phi_n, \tag{4}$$

where $\mathcal{M}(i \to f)$ is the amplitude from the one-particle initial state i with four-momentum $P^\mu = M(1, 0, 0, 0)$ in its rest frame to the n-body final state f. Partial Decay Rate (probability of decay of the particle in a particular decay mode per unit time) is obtained upon phase space integral:

$$\Gamma(i \to f) = \frac{1}{2M} \int |\mathcal{M}(i \to f)|^2 d\Phi_n. \tag{5}$$

Total Decay Rate of a particle:

$$\Gamma_i = \sum_f \Gamma(i \to f), \tag{6}$$

where all possible decay modes are summed up. The lifetime of the particle is given by

$$\tau_i = \frac{1}{\Gamma_i}. \tag{7}$$

One sometimes refers to a “partial lifetime” (a strange terminology),

$$\tau(i \to f) = \frac{1}{\Gamma(i \to f)}. \tag{8}$$

Branching Fraction into a specific decay mode f is given by

$$\mathrm{BR}(i \to f) = \frac{\Gamma(i \to f)}{\Gamma_i} = \frac{\tau_i}{\tau(i \to f)}. \tag{9}$$

From N_0 of the particle i at rest at $t = 0$, the remaining number of the particle at an arbitrary time $t > 0$ is given by

$$N(t) = N_0 e^{-\Gamma_i t} = N_0 e^{-t/\tau_i}. \tag{10}$$

If the particle is moving, there is time dilation effect and the decay is corresondingly delayed: $e^{-\gamma t/\tau_i}$ with $\gamma = E/M$.
Differential Cross Section (two-body to n-body):

$$d\sigma(i \to f) = \frac{1}{2s\bar{\beta}_i}|\mathcal{M}(i \to f)|^2 d\Phi_n, \tag{11}$$

where $s = (k_1 + k_2)^2$, k_1^μ and k_2^μ are four-momenta of initial state particles 1 and 2, $\mathcal{M}(i \to f)$ is the amplitude from the initial state i with particles 1 and 2 to the n-body final state f, and $\bar{\beta}_i$ is defined by

$$\bar{\beta}_i = \sqrt{1 - 2(x_1 + x_2) + (x_1 - x_2)^2} \tag{12}$$

with $x_1 = k_1^2/s = m_1^2/s$, $x_2 = k_2^2/s = m_2^2/s$. $\bar{\beta}_i$ coincides with the velocity of the initial state particles β in the center-of-momentum frame if $m_1 = m_2$. Cross section (number of the scattering events per unit luminosity per unit time) is obtained upon phase space integral:

$$\sigma(i \to f) = \frac{1}{2s\bar{\beta}_i}\int |\mathcal{M}(i \to f)|^2 d\Phi_n. \tag{13}$$

Two-body phase space can be written in a particularly simple manner in the center-of-momentum frame:

$$d\Phi_2 = \frac{\bar{\beta}_f}{8\pi}\frac{d\cos\theta}{2}\frac{d\phi}{2\pi}, \tag{14}$$

where θ, ϕ are polar and azimuthal angles of the momentum $\vec{p}_1$ in the center-of-momentum frame, respectively, and $\bar{\beta}_f$ is the analogous expression of Eq. (12) for the final state particles.