129A HW # 2 (due Sep 19)

1. The momentum is described by the differential operator $\vec{\mathbf{p}} = -i\hbar\vec{\nabla}$ on Schrödinger wave functions $\psi(x)$. (1) Check that the wave function $\psi_p(x) = e^{i\vec{p}\cdot\vec{x}/\hbar}$ is an eigenstate of the momentum $\vec{\mathbf{p}}\psi_p(x) = \vec{p}\psi_p(x)$. Such a wave function is called a plane wave. (2) Using the Schrödinger equation $i\hbar\frac{\partial}{\partial t}\psi_p(x) = \frac{\vec{\mathbf{p}}^2}{2m}\psi_p(x)$, check that the time dependence of this wave function is $\psi_p(x,t) = \psi_p(x,0)e^{-i\vec{p}^2t/2m\hbar} = \exp(i(\vec{p}\cdot\vec{x} - \frac{\vec{p}^2}{2m}t)/\hbar)$. (3) The plane wave is usually written in terms of the frequency ω and the wave vector $\vec{k}$ as $\exp(i(\vec{k}\cdot\vec{x} - \omega t))$. What is the frequency and wave vector of the Schrödinger plane wave? (4) Check that the planes with constant phase factor $(\vec{p}\cdot x - \frac{\vec{p}^2}{2m}t)/\hbar = \text{constant}$ moves with the "phase velocity" $\vec{v}_{phase} = \frac{\vec{p}}{2m}$. This is not a velocity we expect for a particle of mass m and momentum $\vec{p}$. (5) Usually, wave functions are not pure plane waves, but their superpositions (*wave packets*). The peak in a wave packet does not move with the phase velocity but rather with the *group velocity*: $\vec{v}_{group} = \partial\omega/\partial\vec{k}$. Check that this reproduces the expected velocity of the particle.

2. A relativistic generalization of a plane wave is given by $\psi_p(x,t) = e^{i\vec{p}\cdot\vec{x}/\hbar - iEt/\hbar} = e^{-ip^\mu x_\mu/\hbar}$. Note that it has a manifestly Lorentz invariant form (a pleasant surprise!). The only difference from the non-relativistic wave function in **1.** is that we now use $E = \sqrt{(mc^2)^2 + c^2\vec{p}^2}$ instead of $\vec{p}^2/2m$. (1) What is the phase velocity $\vec{v}_{phase}$? Do you notice something strange with it? (2) What is the group velocity $\vec{v}_{group} = \partial E/\partial\vec{p}$? What is its physical meaning?

3. The Lorentz-covariant generalization of angular momenta has six quanitities: $M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$. (1) Check that M^{12} is nothing but the orbital angular momentum L^z. (2) Write down the expression of M^{0i} $(i = 1,2,3)$ for a free particle of mass m and velocity $\vec{v}$. (3) Show that M^{0i} is conserved, *i.e.* $(\partial/\partial t)M^{0i} = 0$. (Hint: $\dot{\vec{v}} = 0$ for a free particle.) (4) For a system of interacting particles with coordinates $\vec{x}_a$, the *sum* $M^{0i}_{tot} = \sum_a M^{0i}_a$ is conserved (independent of time) just like the total energy $E_{tot} = \sum_a E_a$, the total momentum $\vec{P}_{tot} = \sum_a \vec{p}_a$, and the total angular momentum $J^z_{tot} = \sum_a M^{12}_a$ *etc.* Write down the time-dependence of the "center-of-energy" $\vec{X}_{CE} = (\sum_a \vec{x}_a E_a)/E_{tot}$ in terms of the total energy, total momentum and the initial value of M^{0i}_{tot}. (5) What is the non-relativistic limit of $\vec{X}_{CE}$? Neglect $\vec{v}/c$ completely.