Solutions to the Dirac equation

Dirac equation is given by

$$i\hbar\frac{\partial}{\partial t}\psi = c(\alpha\cdot\vec{\mathbf{p}} + mc\beta)\psi, \tag{1}$$

where $\vec{\mathbf{p}} = -i\hbar\vec{\nabla}$ (to be distinguished with c-number $\vec{p}$). Below, we set $\hbar = c = 1$.

First, for a momentum

$$\vec{p} = p(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$

we define two-component eigen-states of the matrix $\vec{\sigma}\cdot\vec{p}$ for later convenience:

$$\chi_+(\vec{p}) = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\phi} \end{pmatrix}, \tag{2}$$

$$\chi_-(\vec{p}) = \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix}, \tag{3}$$

which satisfy

$$(\vec{\sigma}\cdot\vec{p})\chi_\pm(\vec{p}) = \pm p\chi_\pm(\vec{p}). \tag{4}$$

Using $\chi_\pm$, we can write down solutions to the Dirac equation in a simple manner.

Positive energy solutions with momentum $\vec{p}$ have space and time dependence $\psi_\pm(x,t) = u_\pm(p)e^{-iEt+i\vec{p}\cdot\vec{x}}$. The subscript $\pm$ refers to the helicities $\pm1/2$. The Dirac equation then reduces to an equation with no derivatives:

$$E\psi = (\alpha\cdot\vec{p} + m\beta)\psi, \tag{5}$$

where $\vec{p}$ is the momentum vector (not an operator). Explicit solutions can be obtained easily as

$$u_+(p) = \frac{1}{\sqrt{E+m}}\begin{pmatrix} (E+m)\chi_+(\vec{p}) \\ p\chi_+(\vec{p}) \end{pmatrix}, \tag{6}$$

$$u_-(p) = \frac{1}{\sqrt{E+m}}\begin{pmatrix} (E+m)\chi_-(\vec{p}) \\ -p\chi_-(\vec{p}) \end{pmatrix}. \tag{7}$$

Here and below, we adopt normalization $u_\pm^\dagger(p)u_\pm(p) = 2E$ and $E = \sqrt{\vec{p}^2 + m^2}$.

Negative energy solutions must be filled in the vacuum and their "holes" are regarded as anti-particle states. Therefore, it is convenient to assign momentum $-\vec{p}$ and energy $-E = -\sqrt{\vec{p}^2 + m^2}$. The solutions have space and time dependence

$\psi_\pm(x,t) = v_\pm(p)e^{+iEt-i\vec{p}\cdot\vec{x}}$. The Dirac equation again reduces to an equation with no derivatives:

$$-E\psi = (-\alpha\cdot\vec{p} + m\beta)\psi. \tag{8}$$

Explicit solutions are given by

$$v_+(p) = \frac{1}{\sqrt{E+m}}\begin{pmatrix} -p\chi_-(\vec{p}) \\ (E+m)\chi_-(\vec{p}) \end{pmatrix}, \tag{9}$$

$$v_-(p) = \frac{1}{\sqrt{E+m}}\begin{pmatrix} p\chi_+(\vec{p}) \\ (E+m)\chi_+(\vec{p}) \end{pmatrix}. \tag{10}$$

It is convinient to define "barred" spinors $\bar{u} = u^\dagger\gamma^0 = u^\dagger$ and $\bar{v} = v^\dagger\gamma^0$. The γ matrices are defined by

$$\gamma^0 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^i = \beta\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{11}$$

The combination $\bar{u}u$ is a Lorentz-invariant, $\bar{u}u = 2m$, and similarly, $\bar{v}v = -2m$. The combination $\bar{u}\gamma^\mu u$ transforms as a Lorentz vector:

$$\bar{u}_\kappa(p)\gamma^\mu u_\lambda(p) = 2p^\mu\delta_{\kappa,\lambda}, \tag{12}$$

where $\kappa, \lambda = \pm$, and similarly,

$$\bar{v}_\kappa(p)\gamma^\mu v_\lambda(p) = 2p^\mu\delta_{\kappa,\lambda}. \tag{13}$$

They can be interpreted as the "four-current density" which generates electromagnetic field: $\bar{u}\gamma^0 u = u^\dagger u$ is the "charge density," and $\bar{u}\gamma^i u = u^\dagger\alpha^i u$ is the "current density."

Note that the matrix

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{14}$$

commutes with the Hamiltonian in the massless limit $m \to 0$. In fact, at high energies $E \gg m$, the solutions are almost eigenstates of γ_5, with eigenvalues $+1$ for u_+ and v_-, and -1 for u_- and v_+. The eigenvalue of γ_5 is called "chirality." Therefore chirality is a good quantum number in the high energy limit. Neutrinos have chirality minus, and they do not have states with positive chirality.