What’s new with Goldstone?

Haruki Watanabe, Hitoshi Murayama
+ Tomáš Brauner

CERN theory colloquium
December 11, 2013

arXiv:1203.0609, 1302.4800, 1303.1527
all published in PRL
What's wrong with Goldstone?

Haruki Watanabe, Hitoshi Murayama
+ Tomáš Brauner

CERN theory colloquium
December 11, 2013

arXiv:1203.0609, 1302.4800, 1303.1527
all published in PRL
The Nobel Prize in Physics 2013
François Englert, Peter W. Higgs

The Nobel Prize in Physics 2013 was awarded jointly to François Englert and Peter W. Higgs "for the theoretical discovery of a mechanism that contributes to our understanding of the origin of mass of subatomic particles, and which recently was confirmed through the discovery of a predicted fundamental particle, by the ATLAS and CMS experiments at CERN's Large Hadron Collider"
50-year old puzzle

- Goldstone’s theorem
- for every spontaneously broken symmetry, there is a massless excitation
- $E = c \, p$ (linear dispersion relation)
- in Heisenberg ferromagnet
- 1 gapless excitation for 2 broken symm
- $E \propto p^2$ (quadratic dispersion relation)
- What is going on?
Outline

• Introduction (lengthy)
• Main points = simple points
• Formalism –Internal Symmetries–
• Previously known results
• Redundancies (brief)
• Massive NGBs (brief)
• Conclusion
Introduction
SSB is Ubiquitous

- $\chi$-symmetry in strong interactions (QCD)
- crystals (spatial translations)
- $^4$He superfluid, BEC (particle number)
- $^3$He superfluid, spinor BEC, neutron stars, kaon condensation, color superconductivity, etc (a rich variety of symmetries)
- superconductors, Higgs (gauge invariance)
- what is the underlying unified description?
Goldstone's theorem

• When a continuous symmetry $G$ is spontaneously broken to its subgroup $H$, there are massless bosons $E=c \cdot p$ for every broken generators.

$$\langle \pi^a(p)|j^b_\mu(0)|0\rangle = f_\pi \delta^{ab} p^\mu$$

• $n_{\text{NGB}}=n_{\text{BG}}$

• assumes Lorentz invariance and positive definite metric of the Hilbert space
crystal

longitudinal

\[ G = \mathbb{R}^2 \]

\[ H = \mathbb{Z}^2 \]

\[ G/H = \mathbb{T}^2 \]

transverse

\[ \rightarrow \text{phonon, } E = c_{sL} p \]

\[ \rightarrow \text{phonon, } E = c_{sT} p \]
Particle numbers

- U(1) symmetry
  \[ \psi(x) \rightarrow e^{i\theta} \psi(x) \]
  \[ N = \int dx \psi^* \psi \]
- Ginzburg-Landau theory
  \[ V = -\mu \psi^* \psi + \lambda (\psi^* \psi)^2 \]
  \[ \langle 0 | \psi | 0 \rangle \neq 0 \]
- G=U(1), H=0
- \( ^4 \)He superfluid
- scalar BEC
Heisenberg models

\[
H = +J \sum_{\langle i,j \rangle} \vec{s}_i \cdot \vec{s}_j
\]

2 NGBs \( E \propto p \)

\[
H = -J \sum_{\langle i,j \rangle} \vec{s}_i \cdot \vec{s}_j
\]

1 NGB \( E \propto p^2 \)

Both \( G/H = \text{SO}(3)/\text{SO}(2) = S^2 \)
two NGBs? No!

the only mode
experiment

- Dispersion relations can be tested experimentally
- Specific heat
  - \( E \propto p \Rightarrow C_V \propto T^3 \)
  - \( E \propto p^2 \Rightarrow C_V \propto T^{3/2} \)
- Plot \( C_V / T^{3/2} \) vs \( T^{3/2} \)

**FIG. 1.** The specific heat of the two YIG samples. The points for sample 1 give the results obtained in one experiment. The points for sample 2 give the results obtained in two separate experiments.
• SO(3)xU(1) → SO(2)
• $G/H = \mathbb{RP}^3$
• 3 broken generators
• 1 NGB with $E \propto p$
• 1 NGB with $E \propto p^2$
Superconductivity and the associated Nambu-Goldstone modes in a three-flavor atomic Fermi gas with SU(3) symmetry

Lianyi He, Meng Jin, and Pengfei Zhuang
Physics Department, Tsinghua University, Beijing 100084, China
(Received 26 April 2006; published 8 September 2006)

We investigate the superfluidity and the associated Nambu-Goldstone modes in a three-flavor atomic Fermi gas with SU(3) global symmetry. The s-wave pairing occurs in flavor antitriplet channel due to the Pauli principle, and the superfluid state contains both gapped and gapless fermionic excitations. Corresponding to the spontaneous breaking of the SU(3) symmetry to a SU(2) symmetry with five broken generators, there are only three Nambu-Goldstone modes, one is with linear dispersion law and two are with quadratic dispersion law. The other two expected Nambu-Goldstone modes become massive with a mass gap of the order of the fermion energy gap in a wide coupling range. The abnormal number of Nambu-Goldstone modes, the quadratic dispersion law, and the mass gap have significant effect on the low-temperature thermodynamics of the matter.

DOI: 10.1103/PhysRevA.74.033604 PACS number(s): 03.75.Ss, 05.30.Fk, 74.20.Fg, 34.90.+q
Spontaneous Breaking of Lie and Current Algebras

Yoichiro Nambu

Received December 26, 2002; accepted January 29, 2003

The anomalous properties of Nambu–Goldstone bosons, found by Miransky and others in the symmetry breaking induced by a chemical potential, are attributed to the SSB of Lie and current algebras. Ferromagnetism, antiferromagnetism, and their relativistic analogs are discussed as examples.

KEY WORDS: Symmetry breaking; Nambu–Goldstone boson; color superconductivity; chemical potential; ferromagnetism; Lorentz symmetry; current algebra.

1. INTRODUCTION AND SUMMARY

In general the number of the Nambu–Goldstone (NG) bosons associated with a spontaneous symmetry breaking (SSB) $G \rightarrow H$ is equal to the number of symmetry generators $Q_i$ in the coset $G/H$. In the absence of a gauge field, their energy $\omega$ goes as a power $k^\gamma$ of wave number. In a relativistic theory, $\gamma = 1$ necessarily unless Lorentz invariance is broken.

There are, however, exceptions to the above “theorem.”(1–5) Recently
Main points
= simple points

H. Watanabe and HM, arXiv:1203.0609

see also Y. Hidaka, arXiv:1203.1494
Heisenberg models

\[ \mathcal{L}_{\text{eff}} = c_a(\pi)\dot{\pi}^a + \bar{g}_{ab}(\pi)\dot{\pi}^a\dot{\pi}^b - g_{ab}(\pi)\nabla_i \pi^a \nabla_i \pi^b \]

- **anti-ferromagnet**  
  \[ H = +J \sum \vec{s}_i \cdot \vec{s}_j \]  
  \[ \langle 0|J_z^0|0 \rangle = 0 \]  
  2 NGBs  
  \[ E \propto p \]

- **ferromagnet**  
  \[ H = -J \sum \vec{s}_i \cdot \vec{s}_j \]  
  \[ \langle 0|J_z|0 \rangle = -i\langle 0|[J_x, J_y]|0 \rangle \neq 0 \langle i, j \rangle \]  
  1 NGB  
  \[ E \propto p^2 \]

\[ J_x \text{ and } J_y \text{ canonically conjugate to each other cf. } [x, p] = i \hbar \]

describing a single degree of freedom together
General formula

- Define a commutator among broken generators
  \[ \rho_{ab} = \frac{-i}{V} \langle 0 | [Q^a, Q^b] | 0 \rangle \]

- \( n_B = \frac{1}{2} \text{rank} \rho \) counts the number of canonically conjugate pairs (Type-B)
  - each pair describes one d.o.f.
- the remainder \( n_A = n_{BG} - 2n_B \)
- stand-alone NGB d.o.f. (Type-A)

\[ n_{NGB} = n_A + n_B = n_{BG} - \frac{1}{2} \text{rank} \rho \]

conjectured by Watanabe and Brauner

\[ E \propto p^2 \quad \text{generically} \]
\[ E \propto p \quad \text{generically} \]
### Applications

<table>
<thead>
<tr>
<th>example</th>
<th>coset space</th>
<th>BG</th>
<th>NGB</th>
<th>rank ρ</th>
<th>theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>anti-ferromagnet</td>
<td>O(3)/O(2)</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>2=2-0</td>
</tr>
<tr>
<td>ferromagnet</td>
<td>O(3)/O(2)</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1=2-1</td>
</tr>
<tr>
<td>superfluid ⁴He</td>
<td>U(1)</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1=1-0</td>
</tr>
<tr>
<td>superfluid ³He B phase (in magnetic field)</td>
<td>O(3)×O(3)×U(1)/O(2)</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>4=4-0</td>
</tr>
<tr>
<td>BEC (F=0)</td>
<td>U(1)</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1=1-0</td>
</tr>
<tr>
<td>BEC (F=1) polar</td>
<td>O(3)×U(1)/U(1)</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>3=3-0</td>
</tr>
<tr>
<td>BEC (F=1) ferro</td>
<td>O(3)×U(1)/SO(2)</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2=3-1</td>
</tr>
<tr>
<td>3-comp. Fermi liquid</td>
<td>U(3)/U(2)</td>
<td>5</td>
<td>3</td>
<td>4</td>
<td>3=5-2</td>
</tr>
<tr>
<td>neutron star</td>
<td>U(1)</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1=1-0</td>
</tr>
<tr>
<td>kaon cond. (μ=0)</td>
<td>U(2)/U(1)</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>3=3-0</td>
</tr>
<tr>
<td>kaon cond. (μ≠0)</td>
<td>U(2)/U(1)</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2=3-1</td>
</tr>
<tr>
<td>crystal (in magnetic field)</td>
<td>R³/Z³</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>3=3-0</td>
</tr>
</tbody>
</table>

\[ n_{\text{NGB}} = n_{BG} - \text{rank } \rho \]
Formalism
—Internal Symmetries—

H. Watanabe and HM, arXiv:1203.0609
full paper in preparation
Low-$E$ Effective Theory w/ Lorentz-invariance

- Consider $\pi^a(x)$ fields: $\mathbb{R}^{3,1} \rightarrow G/H$ ("pions")
- Write action $S = \int d^4x\ L(\pi, \partial \pi)$ which is $G$-invariant
- Expand in powers of derivative, keep low orders (often up to the second order)
- Lorentz invariance dictates the action to be $S = \int d^4x\ g_{ab}(\pi)\ \partial_\mu \pi^a\ \partial_\mu \pi^b$
- Only data needed is $G$-inv metric on $G/H$
- Indeed, $n_{\text{NGB}} = n_{\text{BG}}$

For $\text{SO}(3)/\text{SO}(2) = S^2$, $S = F^2 \int d^4x\ \partial_\mu n^i\ \partial^\mu n^i$
Low-$E$ Effective Theory w/o Lorentz-invariance

- consider $\pi^a(x)$ fields: $\mathbb{R}^{3,1} \rightarrow G/H$ ("pions")
- Write action $S = \int d^4x \, L(\pi, \partial_t \pi, \partial_x \pi)$ which is $G$-invariant up to a surface term
- expand in powers of derivative, keep low orders (often up to the second order)
- $E = \hbar \omega \propto \partial_t \pi$, $p = \hbar k \propto \partial_x \pi$
- typically up to second powers
- assume translation & rotation inv. of space
non-Lorentz-inv case

\[ \mathcal{L}_{\text{eff}} = g_{ab}(\pi) \partial_\mu \pi^a \partial^\mu \pi^b \]

\[ \mathcal{L}_{\text{eff}} = \bar{g}_{ab}(\pi) \dot{\pi}^a \dot{\pi}^b - g_{ab}(\pi) \nabla_i \pi^a \nabla_i \pi^b \]

- simple generalization to non-Lorentz invariant case, two “metrics” may differ

- in particular, their relative normalization \( c_s^2 \) may not be \( c^2 \)

- more importantly, there is one additional term possible in general (Leutwyler)

\[ \mathcal{L}_{\text{eff}} = c_a(\pi) \dot{\pi}^a + \bar{g}_{ab}(\pi) \dot{\pi}^a \dot{\pi}^b - g_{ab}(\pi) \nabla_i \pi^a \nabla_i \pi^b \]
spectrum

\[ \mathcal{L}_{\text{eff}} = c_a(\pi) \dot{\pi}^a + g_{ab}(\pi) \dot{\pi}^a \dot{\pi}^b - g_{ab}(\pi) \nabla_i \pi^a \nabla_i \pi^b \]

- around the origin \( c_a(\pi) = c_a(0) + \frac{1}{2} c_{ab} \pi^b + O(\pi^2) \)
- in the subspace where \( c_{ab} \) is invertible, \( L = p \dot{q} - H \)
- \( [\pi^a, \pi^b] = -i (c^{-1})^{ab} \)
- the \( c_a \) term dominates over \( g_{ab} \) term \( E \gg E^2 \)
- broken Noether currents \( J^0_a = \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \dot{\pi}^b} \delta_a \pi^b = c_{ba} \pi^b \)
- \( [J^0_a, Q_b] = -i c_{ca} c_{db} (c^{-1})_{cd} = i c_{ab} \)
- Namely, for \( \rho_{ab} = \frac{-i}{V} \langle 0 | [Q^a, Q^b] | 0 \rangle \) \( c_a \dot{\pi}^a \approx \frac{1}{2} \rho_{ab} \pi^b \dot{\pi}^a \)
- when \( c_a \) present, \( E \propto p^2 \), otherwise \( E \propto p \)!
- \( \Pi^a, \Pi^b \) canonically conjugate, describe 1 dof.
Bottomline

\[ \mathcal{L}_{\text{eff}} = c_a (\pi) \dot{\pi}^a + \bar{g}_{ab} (\pi) \dot{\pi}^a \dot{\pi}^b - g_{ab} (\pi) \nabla_i \pi^a \nabla_i \pi^b \]

- SSB leads to gapless excitations (NGBs)
- Lorentz invariance: \( n_{\text{NGB}} = n_{\text{BG}} \), \( E = c \rho \)
- w/o Lorentz invariance:
  - Type A: \( \rho_{ab} = 0 \), \( E \propto p \)
  - Type B: \( \rho_{ab} \neq 0 \), \( E \propto p^2 \)
    - \( n_{\text{NGB}} = n_A + n_B \)
    - \( n_{\text{BG}} = n_A + 2n_B \)
- explicit effective Lagrangian \( \implies \) interactions
- underlying partially symplectic geometry

For \( \text{SO}(3)/\text{SO}(2) = S^2 \),

\[ \mathcal{L}_{\text{eff}} = \frac{n_x \dot{n}_y - n_y \dot{n}_x}{1 + n_z} - \vec{\nabla} n_i \vec{\nabla} n_i \]
geometry

\[ \mathcal{L}_{\text{eff}} = c_a(\pi) \dot{\pi}^a + \bar{g}_{ab}(\pi) \dot{\pi}^a \dot{\pi}^b - g_{ab}(\pi) \nabla_i \pi^a \nabla_i \pi^b \]

- What is \( c_a(\pi) \)?
- It defines one-form \( c_1 = c_a(\pi) d\pi^a \) on \( G/H \)
- \( L \) must be \( G \)-invariant up to a surface term
  \[ \mathcal{L}_{V_i} c_1 = d\chi \]
- Its exterior derivative is \( G \)-invariant
  \[ \omega_2 = d c_1 \]
  \[ \mathcal{L}_{V_i} \omega_2 = d \mathcal{L}_{V_i} c_1 = d^2 \chi = 0 \]
- Namely, \( G/H \) is endowed with a \( G \)-invariant closed two-form \( \omega_2 \) (may be degenerate)

Darboux’s theorem:
\[ \omega_2 = \sum dp_i \wedge dq_i \]

presymplectic structure
geometry

\[ \mathcal{L}_{\text{eff}} = c_a(\pi) \dot{\pi}^a + \bar{g}_{ab}(\pi) \dot{\pi}^a \dot{\pi}^b - g_{ab}(\pi) \nabla_i \pi^a \nabla_i \pi^b \]

- What is \( c_a(\pi) \)?
- It defines one-form \( c_1 = c_a(\pi) \, d\pi^a \) on \( G/H \)
- \( L \) must be \( G \)-invariant up to a surface term
  \[ \mathcal{L}_{V_i} c = i_{V_i} dc + d(i_{V_i} c) = [de_i] + d(i_{V_i} c) \quad \text{de}_i = i_{V_i} \omega \]
- The Noether current picks up surface term
  \[ j^0_i = -\bar{g}_{ab} h_i^a \dot{\pi}^b + e_i \]
- In the ground state = stationary:
  \[ \langle 0 | j^0_i | 0 \rangle = e_i(0) \]
- It is "charge density" of the ground state
General Geometry

NGBs for generators $a$ and $b$ are symplectic pairs and describe a single degree of freedom

$$\dim G - \dim H = n_A + 2n_B$$

projection possible for compact semi-simple
explicit construction

- for compact semi-simple case, we found closed expressions

\[ gU = U' h'(\pi', g) \]
\[ \omega = U^{-1} dU = \sum T_k \omega^k \]
\[ g_{ab}(\pi) = g_{cd}(0) \omega^c_{\alpha}(\pi) \omega^d_{\beta}(\pi) \]
\[ \langle 0 | j_i^0 | 0 \rangle = e_i(0) \]
\[ c = -\omega^k e_k(0) \]
central extension

$$\mathcal{L}_{\text{eff}} = c_a(\pi)\dot{\pi}^a + \bar{g}_{ab}(\pi)\dot{\pi}^a\dot{\pi}^b - g_{ab}(\pi)\nabla_i\pi^a\nabla_i\pi^b$$

$$de_i = i_{V_i}\omega$$

$$\mathcal{L}_{V_j}de_j = \mathcal{L}_{V_j}i_{V_i}\omega = i[V_j,V_i]\omega + i_{V_i}\mathcal{L}_{V_j}\omega = f_{ji}^k i_{V_k}\omega = f_{ji}^k de_k$$

$$\mathcal{L}_{V_j}e_j = f_{ji}^k e_k + z_{ji}$$

- If $H^2(\mathfrak{g}) \neq 0$, a central extension $z_{ji} \neq 0$ possible
- impossible for semi-simple $\mathfrak{g}$
- possible for multiple U(1)'s, $\mathbb{R}$'s
- important when magnetic field
Examples: $n_{BG} = 1$

$$\mathcal{L}_{\text{eff}} = c_a (\pi) \dot{\pi}^a + \bar{g}_{ab} (\pi) \dot{\pi}^a \dot{\pi}^b - g_{ab} (\pi) \nabla_i \pi^a \nabla_i \pi^b$$

- spontaneously broken U(1): scalar BEC
  $$\mathcal{L} = \frac{1}{2} \dot{\theta}^2 - \frac{1}{2} c_s^2 (\vec{\nabla} \theta)^2$$

- the only difference from Lorentz-invariant case is the metric can have different normalization for space and time

- it is the speed of sound
Examples: $n_{BG}=2$

\[ \mathcal{L}_{\text{eff}} = c_a(\pi) \dot{\pi}^a + \bar{g}_{ab}(\pi) \dot{\pi}^a \dot{\pi}^b - g_{ab}(\pi) \nabla_i \pi^a \nabla_i \pi^b \]

- $\mathbb{R}^2$: invariant closed two-form is
  \[ \omega_2 = dx \wedge dy = \frac{i}{2} dz \wedge d\bar{z} \]
  \[ c_1 = \frac{i}{2} zd\bar{z} \]

- the leading terms are
  \[ \mathcal{L}_{\text{eff}} = \frac{i}{2} \dot{z} \dot{\bar{z}} - \frac{1}{2m} \nabla z \nabla \bar{z} \]

- free non-rel particle with one dof, $E \propto p^2$

- or 2d lattice in $B$, with one dof, $E \propto p^2$

\[ [z(x), \bar{z}(y)] = -i \delta(x - y) \] central extension $H^2(\mathbb{R}^2) \neq 0$
Examples: $n_{\text{BG}}=2$

\[ \mathcal{L}_{\text{eff}} = c_a(\pi) \dot{\pi}^a + \bar{g}_{ab}(\pi) \dot{\pi}^a \dot{\pi}^b - g_{ab}(\pi) \nabla_i \pi^a \nabla_i \pi^b \]

- $\mathbb{R}^2$: invariant closed two-form is
  \[ \omega_2 = dx \wedge dy = \frac{i}{2} dz \wedge d\bar{z} \]
  \[ c_1 = \frac{i}{2} zd\bar{z} \]

- But if $c_1$ absent, need to consider 2nd term
  \[ \mathcal{L}_{\text{eff}} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2} c_s^2((\vec{\nabla} x)^2 + (\vec{\nabla} y)^2) \]

- e.g., 2D lattice with two dof, $E \propto p$
Examples: $n_{BG}=2$

$$\mathcal{L}_{\text{eff}} = c_a (\pi) \dot{\pi}^a + \bar{g}_{ab} (\pi) \dot{\pi}^a \dot{\pi}^b - g_{ab} (\pi) \nabla_i \pi^a \nabla_i \pi^b$$

- $S^2$: $\text{SO}(3)$-invariant closed two-form is
  $$\omega_2 = d(\cos \theta) \wedge d\phi$$
  $$dc_1 = \frac{1}{2} d \frac{n_y dn_x - n_x dn_y}{1 + n_z} = d[(-1 + \cos \theta) d\phi]$$

- the leading terms are
  $$\mathcal{L}_{\text{eff}} = \frac{1}{2} \frac{n_y \dot{n}_x - n_x \dot{n}_y}{1 + n_z} - c_s^2 \frac{1}{2} \vec{\nabla} n_i \vec{\nabla} n_i$$

- ferromagnet with one dof, $E \propto p^2$
Examples: $n_{BG}=2$

$$\mathcal{L}_{\text{eff}} = c_a (\pi) \dot{\pi}^a + \bar{g}_{ab} (\pi) \dot{\pi}^a \dot{\pi}^b - g_{ab} (\pi) \nabla_i \pi^a \nabla_i \pi^b$$

- $S^2$: $SO(3)$-invariant closed two-form is

$$\omega_2 = d(\cos \theta) \wedge d\phi$$

$$dc_1 = \frac{1}{2} d \frac{n_y dn_x - n_x dn_y}{1 + n_z} = d[(-1 + \cos \theta) d\phi]$$

- But if $c_1$ absent, need to consider 2nd term

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} \dot{n}_i \dot{n}_i - c_s^2 \frac{1}{2} \vec{\nabla} n_i \vec{\nabla} n_i$$

- anti-ferromagnet with two dof, $E \propto p$
Examples: $n_{BG}=3$

$$\mathcal{L}_{\text{eff}} = c_a(\pi)\dot{\pi}^a + \bar{g}_{ab}(\pi)\dot{\pi}^a\dot{\pi}^b - g_{ab}(\pi)\nabla_i\pi^a\nabla_i\pi^b$$

• $\text{SO}(3) \times \text{U}(1)/\text{SO}(2) = \mathbb{R}P^3 = S^3/\mathbb{Z}_2$

• spinor BEC ferromagnetic phase:

$$\psi = v \frac{e^{i\theta}}{\sqrt{2(1 + \bar{z}z)}} \begin{pmatrix} 1 - z^2 \\ i(1 + z^2) \\ 2z \end{pmatrix} \quad \psi^\dagger \psi = v^2$$

$$\psi^*i\dot{\psi} = v^2 \left( -\dot{\theta} + i \frac{\dot{z}^*\dot{z} - \dot{z}^*\dot{z}}{1 + z^*z} \right)$$

• one dof with $E \propto p^2$, one dof with $E \propto p$

Hopf map $\mathbb{R}P^3 \to S^2$ down to a symplectic homogeneous $S^2$
Examples: $n_{BG}=3$

$$\mathcal{L}_{\text{eff}} = c_a(\pi) \dot{\pi}^a + \bar{g}_{ab}(\pi) \dot{\pi}^a \dot{\pi}^b - g_{ab}(\pi) \nabla_i \pi^a \nabla_i \pi^b$$

- $\text{SO}(3) \times \text{U}(1)/\text{SO}(2) = (S^2 \times S^1)/\mathbb{Z}_2$

- spinor BEC polar phase:

$$\psi = ve^{i\theta} \vec{n} \quad \vec{n}^2 = 1$$

$$\psi^* i \dot{\psi} = v^2 i \dot{\theta} \approx 0$$

- three dof with $E \propto p$

- vanishing presymplectic structure
Examples: $n_{BG}=3$

\[ \mathcal{L}_{\text{eff}} = c_a(\pi)\dot{\pi}^a + \bar{g}_{ab}(\pi)\dot{\pi}^a\dot{\pi}^b - g_{ab}(\pi)\nabla_i\pi^a\nabla_i\pi^b \]

• $U(2)/U(1)=S^3$ (kaon condensation):

\[
\psi = v \frac{e^{i\theta}}{\sqrt{1 + \bar{z}z}} \begin{pmatrix} 1 \\ z \end{pmatrix} \quad \psi^\dagger\psi = v^2
\]

\[
i\psi^\dagger\dot{\psi} = -\dot{\theta} + i\frac{1}{2} \frac{\bar{z}\dot{z} - \dot{\bar{z}}z}{1 + \bar{z}z}
\]

chemical potential

• the leading terms are

\[
\mathcal{L}_{\text{eff}} = +i\frac{1}{2} \frac{\bar{z}\dot{z} - \dot{\bar{z}}z}{1 + \bar{z}z} + \left( \dot{\theta} - i\frac{1}{2} \frac{\bar{z}\dot{z} - \dot{\bar{z}}z}{1 + \bar{z}z} \right)^2
\]

\[
- \left[ \left( \vec{\nabla}\theta - i\frac{1}{2} \frac{\bar{z}\vec{\nabla}z - \vec{\nabla}\bar{z}z}{1 + \bar{z}z} \right)^2 + 2\frac{\vec{\nabla}\bar{z}\vec{\nabla}z}{(1 + \bar{z}z)^2} \right]
\]

• one dof with $E \propto p^2$, one dof with $E \propto p$

Hopf map $S^3 \rightarrow S^2$ down to a symplectic homogeneous $S^2$
NGBs for generators $a$ and $b$ are symplectic pairs and describe a single degree of freedom

$$\dim G - \dim H = n_A + 2n_B$$
### Applications

<table>
<thead>
<tr>
<th>example</th>
<th>coset space</th>
<th>BG</th>
<th>NGB</th>
<th>rank_ρ</th>
<th>theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>anti-ferromagnet</td>
<td>$O(3)/O(2)$</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>$2=2-0$</td>
</tr>
<tr>
<td>ferromagnet</td>
<td>$O(3)/O(2)$</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>$1=2-1$</td>
</tr>
<tr>
<td>superfluid $^4$He</td>
<td>$U(1)$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$1=1-0$</td>
</tr>
<tr>
<td>superfluid $^3$He B phase (in magnetic field)</td>
<td>$O(3)\times O(3)\times U(1)/O(2)$</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>$4=4-0$</td>
</tr>
<tr>
<td>BEC (F=0)</td>
<td>$U(1)$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$1=1-0$</td>
</tr>
<tr>
<td>BEC (F=1) polar</td>
<td>$O(3)\times U(1)/U(1)$</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>$3=3-0$</td>
</tr>
<tr>
<td>BEC (F=1) ferro</td>
<td>$O(3)\times U(1)/SO(2)$</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>$2=3-1$</td>
</tr>
<tr>
<td>3-comp. Fermi liquid</td>
<td>$U(3)/U(2)$</td>
<td>5</td>
<td>3</td>
<td>4</td>
<td>$3=5-2$</td>
</tr>
<tr>
<td>neutron star</td>
<td>$U(1)$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$1=1-0$</td>
</tr>
<tr>
<td>kaon cond. ($\mu=0$)</td>
<td>$U(2)/U(1)$</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>$3=3-0$</td>
</tr>
<tr>
<td>kaon cond. ($\mu \neq 0$)</td>
<td>$U(2)/U(1)$</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>$2=3-1$</td>
</tr>
<tr>
<td>crystal (in magnetic field)</td>
<td>$R^3/Z^3$</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>$3=3-0$</td>
</tr>
</tbody>
</table>

$n_{NGB} = n_{BG}$
stability\(\text{@} T=0\) in \(d+1\) dim

- **Type A:**
  - scaling
  - interaction
  - IR free for \(d\geq2\) \((d=1\) symmetry restored\)

- **Type B:**
  - scaling
  - interaction
  - IR free for \(d\geq1\)

\[
\mathcal{L}_{\text{eff}} = \bar{g}_{ab} \dot{\pi}^a \dot{\pi}^b - g_{ab} \nabla_i \pi^a \nabla \pi^b
\]

\[
\bar{x}' = a \bar{x}, \quad t' = at
\]

\[
\pi'^a(a \bar{x}, at) = a^{(1-d)/2} \pi^a(\bar{x}, t)
\]

\[
\nabla_i \pi^a \nabla_i \pi^b \pi^c \sim a^{-(d-1)/2}
\]

\[
\mathcal{L}_{\text{eff}} = \rho_{ab} \pi^a \dot{\pi}^b - g_{ab} \nabla_i \pi^a \nabla \pi^b
\]

\[
\bar{x}' = a \bar{x}, \quad t' = a^2 t
\]

\[
\pi'^a(a \bar{x}, a^2 t) = a^{-d/2} \pi^a(\bar{x}, t)
\]

\[
\nabla_i \pi^a \nabla_i \pi^b \pi^c \sim a^{-d/2}
\]

\(\text{à la Hořava-Lifshitz}\)
Previously Known Theorems
Nielsen-Chadha theorem

\[ \mathcal{L}_{\text{eff}} = c_a(\pi)\dot{\pi}^a + g_{ab}(\pi)\dot{\pi}^a \dot{\pi}^b - g_{ab}(\pi)\nabla_i \pi^a \nabla_i \pi^b \]

- Type-I if \( E \propto p^{2n+1} \)
- Type-II if \( E \propto p^{2n} \)
- Proved \( n_I + 2n_{II} \geq n_{BG} \)
- only an inequality, a weak statement
- follows from our result \( n_A + 2n_B = n_{BG} \) because Type-A (B) is generically Type-I (II)
- but not the same if \( O(\nabla^2) \) term absent and \( L \) starts with \( O(\nabla^4) \), then Type-A but Type-II
Schäfer et al theorem

\[ \mathcal{L}_{\text{eff}} = c_a (\pi) \dot{\pi}^a + \bar{g}_{ab} (\pi) \dot{\pi}^a \dot{\pi}^b - g_{ab} (\pi) \nabla_i \pi^a \nabla_i \pi^b \]

\[ c_a \dot{\pi}^a \approx \frac{1}{2} \rho_{ab} \pi^b \dot{\pi}^a \]

- If \( \rho_{ab} = -i \langle 0 | [Q_a, Q_b] | 0 \rangle / V = 0 \)
- then no Type-B
- \( n_{\text{NGB}} = n_{\text{BG}} = n_A \)

\[ n_{\text{NGB}} = n_{\text{BG}} - \frac{1}{2} \text{rank} \rho \]

conjectured by Watanabe and Brauner
no-go case

- Not every NGBs can be paired as Type-B
- SU(3)/U(1)^2: Kähler and symplectic

<table>
<thead>
<tr>
<th>Type-A</th>
<th>Type-B</th>
<th>$n_A+2n_B=6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>
redundancies

H. Watanabe and HM, arXiv:1203.0609
another one in preparation w/ T. Brauner
spacetime symmetries

- so far all discussions are internal symmetries
- but there are situations when $n_{NGB}$ is further reduced for spacetime symmetries
- spontaneously broken scale and conformal symmetries lead to only one NGB (dilaton) (Salam-Strathdee)
- crystal breaks both translations ($P_i$) and rotations ($J_i$), but only phonons for $P_i$
Noether constraints

- They can be understood as a consequence of Noether constraints
  \[ \int d^d x \sum_a c_a(x) j_a^0(x) |0\rangle = 0 \]
- For broken symmetries, we have \[ \langle \pi_b | j_a^0(x) |0\rangle \neq 0 \]
- Then they are linearly redundant

\[
0 = \sum_b |\pi_b\rangle \langle \pi_b | \int d^d x \sum_a c_a(x) j_a^0(x) |0\rangle \\
= \sum_b |\pi_b\rangle \int d^d x c_a(x) \langle \pi_b | j_b^0(x) |0\rangle
\]
Examples

• crystal: translations and rotations are both spontaneously broken

• they are both generated by the energy-momentum tensor \( R^{0i} = \epsilon_{ijk} x^j T^{0k} \)

• would-be NGBs for rotations are the same excitations as those for translations (phonons)
Examples

- Ginzburg-Landau theory
  \[ V = -\mu \psi^* \psi + \lambda (\psi^* \psi)^2 \]
- \(G=U(1), H=0\)
- \(^4\text{He} \) superfluid
- scalar BEC \( \langle 0|\psi|0 \rangle \neq 0 \)
- \(U(1)\) \(\psi(\vec{x}, t) \rightarrow e^{i\theta} \psi(\vec{x}, t)\)
- Galilean boost
  \(\psi(\vec{x}, t) \rightarrow e^{i(m \vec{x} \cdot \vec{x} - \frac{1}{2} m \vec{v}^2 t)} \psi(\vec{x} - \vec{v} t, t)\)
- both broken \(n_{BG}=1+3=4\)

\[ B^{i\mu} = t T^{i\mu} - m x^i j^\mu \]

\(\Rightarrow\) no separate NGBs for Galilean boosts
vortex lattice

- rotate a (2d) BEC
- vortices form a triangular lattice
- broken: $U(1)$, $P_{x,y}, J_z$
- only one Type-A NGB with
  \[ E \propto p^2 \]
- called Tkachenko mode
  \[ T^{0i} = mj^i - 2m\Omega \epsilon^{ij} x^j j^0 \]

we have a precise effective Lagrangian for this
relation to inverse Higgs mechanism

• old idea called “inverse Higgs mechanism” is used to eliminate spurious NGBs in case of spacetime symmetries E.A. Ivanov and V.I. Ogievetskiii, 1975
• not much discussed in cases without Lorentz invariance
• not applicable when translation is broken
• recently more papers
  • Endlich, Nicolis, Penco, arXiv:1310.2272
  • Hayata, Hidaka, arXiv:1312.0008
massive NGB

H. Watanabe, T. Brauner, and HM, arXiv:1303.1527
Nicolis-Piazza

- normally, we can say few things about gapped modes based on symmetries alone (cf. BPS)

- They pointed out in Lorentz-invariant systems and broken symmetries, some gaps can be predicted exactly with group theory

\[ \tilde{H} = H - \mu Q \]
\[ [Q, E_{\pm\alpha}] = \pm \alpha E_{\pm\alpha} \]
\[ \tilde{H}(E_{\alpha}|0\rangle) = \mu \alpha (E_{\alpha}|0\rangle) \]

but not for the conjugate generator
massive NGB

- It turns out the system does not need to be Lorentz invariant, nor $Q$ broken

- quite general result applicable in many systems

\[ n_{mNGB} = \frac{1}{2} (\text{rank} \rho - \text{rank} \tilde{\rho}) \]

\[ \rho_{ab} = \frac{-i}{V} \langle 0 | [Q_a, Q_b] | 0 \rangle \quad [Q_a, H] = 0 \]

\[ \tilde{\rho}_{ab} = \frac{-i}{V} \langle 0 | [\tilde{Q}_a, \tilde{Q}_b] | 0 \rangle \quad [\tilde{Q}_a, \tilde{H}] = [\tilde{Q}_a, H - \mu Q] = 0 \]

\[ \tilde{H}(E_\alpha | 0 \rangle) = \mu \alpha (E_\alpha | 0 \rangle) \]
Examples

• ferromagnet and anti-ferromagnet in a constant magnetic field

• relativistic BECs, kaon condensation

• QCD with chemical potential for isospin

• many examples previously known based on approximation methods, now are exact
Conclusion

• age-old subject, yet still a lot to learn!
• for internal symmetries, precise counting rule and dispersion relation of NGBs finally known
• underlying geometry: presymplectic structure
• redundancies make sense for spacetime symmetries
• exact predictions on massive NGBs à la BPS
Field Theories with «Superconductor» Solutions.

J. GOLDSTONE

CERN - Geneva

(ricevuto 1° Settembre 1960)

Summary. — The conditions for the existence of non-perturbative type «superconductor» solutions of field theories are examined. A non-covariant canonical transformation method is used to find such solutions for a theory of a fermion interacting with a pseudoscalar boson. A covariant renormalisable method using Feynman integrals is then given. A «superconductor» solution is found whenever in the normal perturbative-type solution the boson mass squared is negative and the coupling constants satisfy certain inequalities. The symmetry properties of such solutions are examined with the aid of a simple model of self-interacting boson fields. The solutions have lower symmetry than the Lagrangian, and contain mass zero bosons.
pathology

• In the presence of central extension, there are examples where the presymplectic structure cannot be projected down to a symplectic homogeneous space

• $T^3$ with $\omega = d\theta^1 \wedge (d\theta^2 + rd\theta^3)$

• if $r$ irrational, the projection would be dense and ill-defined

• I consider such a case pathological