233B HW #1

1. Synchrotron radiation loss

According to the formula (1), the synchrotron radiation loss scales as E^4 / R^2, once $v \approx c$. However, in order to compensate for the falling cross section $\sigma \propto 1 / E^2$, we need to increase $\mathcal{L} \propto E^2$, which requires $N_\pm \propto E$ according to Eq. (2) if f_c is kept constant. Therefore, the overall synchrotron loss scales as E^5 / R^2 to maintain the same event rate for annihilation processes. Of course, one could have chosen to increase the repetition rate $f_c \propto E^2$, but that would require the number of bunches to increase as E^2, and the synchrotron radiation loss as E^6 / R^2. Therefore, increasing the number of electrons in the beam is more benefitial (if it is possible to maintain the beam stable is an entirely different manner). Using $R = 27\,\text{km}$ for $E = 100$ GeV ($\sqrt{s} = 200$ GeV for LEP-II), we scale it up to $R = 27\,\text{km} * (2\,\text{TeV} / 200\,\text{GeV})^{5/2} = 8500\text{km}$. This is clearly impractical.

There is currently only one known way to overcome this problem in designing higher energy $e^+ e^-$ colliders. You make it completely linear, with two beams colliding head-on from two LINACs (LINear ACcelerators). This is the concept behind the ILC (International Linear Collider).

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N[27 * 10^(5/2)]
8538.15
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2. α_s from J/ψ decay

According to the PDG particle listing, the branching fractions of J/ψ decays are

inclusive hadronic: $87.7 \pm 0.5\,\%$

among which virtual photon contribution is $13.50 \pm 0.30\,\%$

$\mu^+ \mu^-$: $5.93 \pm 0.06\,\%$

Separating out the $g\,g\,g$ contribution (non-virtual-photon piece), we find $74.2 \pm 0.6\,\%$ (assumed uncorrelated Gaussian error, which may not be too conservative.) Taking the ratio to $\mu^+ \mu^-$, we find $\frac{\Gamma(J/\psi \to g\,g\,g)}{\Gamma(J/\psi \to \mu^+ \mu^-)} = 12.5 \pm 0.16$. Equating it to the lowest order prediction, we find $\alpha_s = 0.205 \pm 0.001$. The error due to the higher order corrections is actually much bigger than this error bar which we ignore. (Actually at this energy, α is more like $1/134$, which changes the result very little.) This result is not crazy (but a little low), compared to Fig. 9.2 in Ian Hinchliffe's review on QCD in PDG, http://pdg.lbl.gov/2007/reviews/qcdrpp.pdf

As a cross check, we can see if the hadronic width through the virtual photon agrees with the lowest order formula. We find $\frac{\Gamma(J/\psi \to q\,\bar{q})}{\Gamma(J/\psi \to \mu^+ \mu^-)} = 2.28$, which should be compared to $N_c\left(\left(\frac{2}{3}\right)^2 + \left(\frac{-1}{3}\right)^2 + \left(\frac{-1}{3}\right)^2\right) = 2$, which is in a *reasonably* good agreement given that we ignore the higher order QCD corrections. We see that there are 10-20% errors because we ignored the higher order corrections.

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87.7 - 13.5
74.2
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Sqrt[0.5^2 + 0.3^2]
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0.583095

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74.2 / 5.93
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12.5126

$$\% * \text{Sqrt}\left[\left(\frac{0.6}{74.2}\right)^2 + \left(\frac{0.06}{5.93}\right)^2\right]$$

0.162068

$$\left(\frac{16\,\pi\,\left(\frac{2}{3}\right)^2 \left(\frac{1}{137}\right)^2}{\frac{160}{81}\left(\pi^2 - 9\right)}\,12.51\right)^{1/3}$$

0.205423

$$\frac{13.50}{5.93}$$

2.27656

3. J/ψ line shape

The cross section into $g\,g\,g$ final state is given by the Breit-Wigner formula because there is no interference, while that into $q\,\overline{q}$ final state interfers between the resonant and virtual photon contributions. The Breit-Wigner contribution is

$\sigma_{\text{BW}} = \frac{2J+1}{(2S_1+1)(2S_2+1)}\,\frac{4\pi}{k^2}\,\frac{m^2\Gamma^2}{\left(s-m^2\right)^2+m^2\Gamma^2}\,B_{\text{in}}\,B_{\text{out}} = \frac{12\pi}{s}\,\frac{m^2\Gamma^2}{\left(s-m^2\right)^2+m^2\Gamma^2}\,B_{\text{in}}\,B_{\text{out}}.$

The case of the $\mu^+\,\mu^-$ final state has the interference and the cross section is

$\sigma = \frac{4\pi\alpha^2}{3s}\left|1 + \frac{\xi^2 s}{s-m^2+i\,m\,\Gamma}\right|^2$

To fix ξ, we drop the non-resonant piece and require it gives the Breit-Wigner formula,

$\frac{4\pi\alpha^2}{3s}\,\frac{\xi^4 s^2}{\left(s-m^2\right)^2+m^2\Gamma^2} = \frac{12\pi}{s}\,\frac{m^2\Gamma^2}{\left(s-m^2\right)^2+m^2\Gamma^2}\,B_{\text{in}}\,B_{\text{out}}$

and hence (using $s \approx m^2$),

$\alpha^2\,\xi^4\,m^2 = 9\,\Gamma^2\,B_{e\,e}\,B_{\mu\,\mu}$

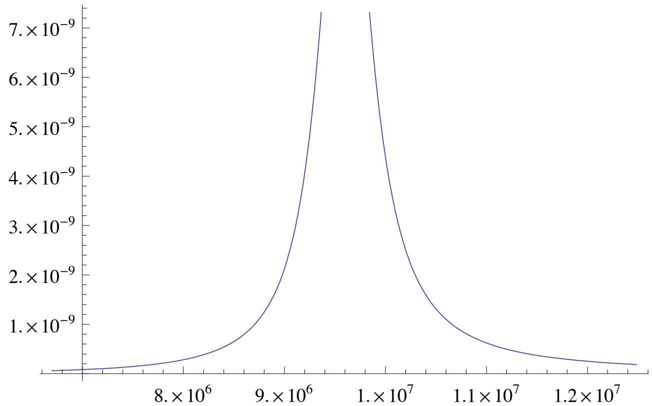
or

$\alpha\,\xi^2\,m = 3\,\Gamma\,B_{l\,l}$

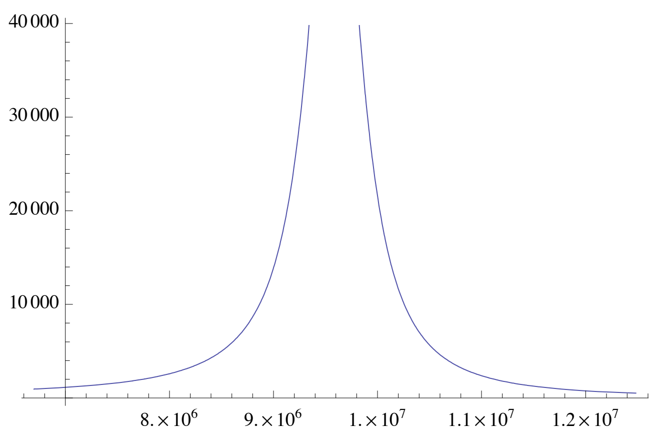
Having fixed ξ, the cross section to the $q\,\overline{q}$ final state is

$\sigma = \frac{4\pi\alpha^2}{3s}\left|1 + \frac{3\,m\,\Gamma\,B_{l\,l}}{\alpha}\,\frac{1}{s-m^2+i\,m\,\Gamma}\right|^2\frac{B_{q\,q}}{B_{l\,l}}$

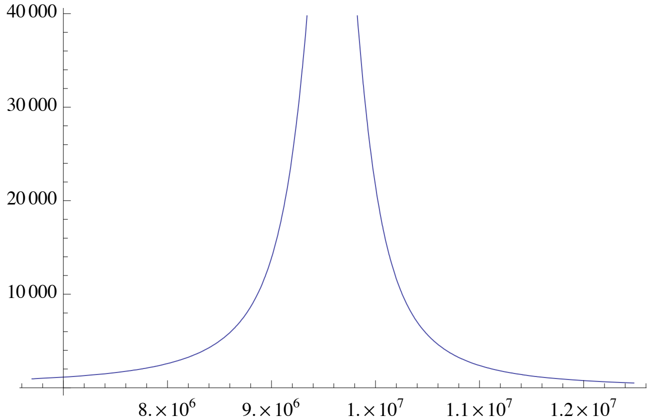
```
Plot[86.8/(s 10^6) Abs[1 + (3 m Γ B11)/α 1/(s - m^2 + I m Γ)]^2 Bqq/B11 /.
  {m -> 3096.916, Γ → 93.4, α → 1/137, B11 → 0.0594, Bqq → 0.1350},
 {s, 3096.916^2 - 10×3096.916×93.4, 3096.916^2 + 10×3096.916×93.4}]
```




```
Plot[0.389 10^6 (12 π)/(s 10^-6) Abs[(m Γ)/(s - m^2 + I m Γ)]^2 Bggg B11 /.
  {m -> 3096.916, Γ → 93.4, α → 1/137, B11 → 0.0594, Bqq → 0.1350, Bggg → 0.877 - 0.1350},
 {s, 3096.916^2 - 10×3096.916×93.4, 3096.916^2 + 10×3096.916×93.4}]
```



```
Plot[86.8/(s 10^-6) Abs[(3 m Γ B11)/α 1/(s - m^2 + I m Γ)]^2 Bggg/B11 /.
  {m -> 3096.916, Γ → 93.4, α → 1/137, B11 → 0.0594, Bqq → 0.1350, Bggg → 0.877 - 0.1350},
 {s, 3096.916^2 - 10 × 3096.916 × 93.4, 3096.916^2 + 10 × 3096.916 × 93.4}]
```

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Show[%, %%]
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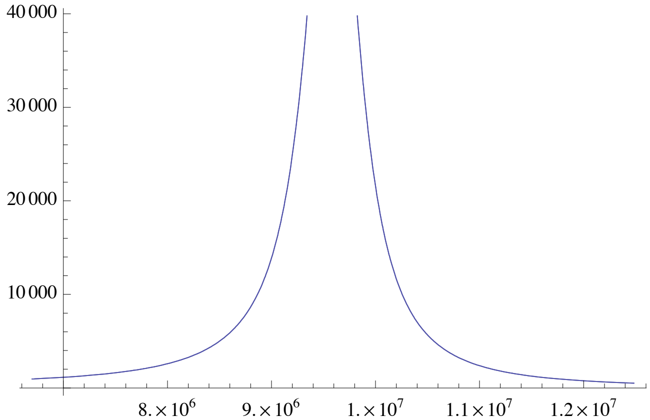
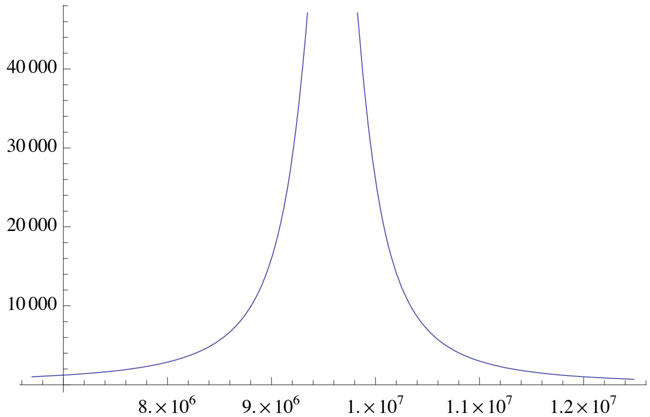


```
Plot[ 86.8/(s 10^-6) Abs[1 + (3 m Γ B11)/α 1/(s - m^2 + I m Γ)]^2 Bqq/B11 + 86.8/(s 10^-6) Abs[(3 m Γ B11)/α 1/(s - m^2 + I m Γ)]^2 Bggg/B11 /.
  {m -> 3096.916, Γ → 93.4, α → 1/137, B11 → 0.0594, Bqq → 0.1350, Bggg → 0.877 - 0.1350},
  {s, 3096.916^2 - 10 × 3096.916 × 93.4, 3096.916^2 + 10 × 3096.916 × 93.4}]
```

$$\text{Plot}\left[\frac{86.8}{s\,10^{-6}}\,\text{Abs}\left[1+\frac{3\,m\,\Gamma\,B_{11}}{\alpha}\,\frac{1}{s-m^2+I\,m\,\Gamma}\right]^2\frac{B_{qq}}{B_{11}}+\frac{86.8}{s\,10^{-6}}\,\text{Abs}\left[\frac{3\,m\,\Gamma\,B_{11}}{\alpha}\,\frac{1}{s-m^2+I\,m\,\Gamma}\right]^2\frac{B_{ggg}}{B_{11}}\;/.\right.$$
$$\left\{m\to 3096.916,\ \Gamma\to 93.4,\ \alpha\to\frac{1}{137},\ B_{11}\to 0.0594,\ B_{qq}\to 0.1350,\ B_{ggg}\to 0.877-0.1350\right\},$$
$$\left.\left\{s,\ 3096.916^2-10\times 3096.916\times 93.4,\ 3096.916^2+10\times 3096.916\times 93.4\right\}\right]$$



$$\text{Plot}\left[\text{NIntegrate}\left[\frac{1}{\sqrt{2\pi}\,\sigma}\,E^{-\left(\sqrt{s}-t\right)^2/2/\sigma^2}\,\frac{86.8}{s\,10^{-6}}\,\text{Abs}\left[1+\frac{3\,m\,\Gamma\,B_{11}}{\alpha}\,\frac{1}{s-m^2+I\,m\,\Gamma}\right]^2\frac{B_{qq}}{B_{11}}+\frac{86.8}{s\,10^{-6}}\,\text{Abs}\left[\frac{3\,m\,\Gamma\,B_{11}}{\alpha}\,\frac{1}{s-m^2+I\,m\,\Gamma}\right]^2\right.\right.$$
$$\frac{B_{ggg}}{B_{11}}\;/.\;\left\{m\to 3096.916,\ \Gamma\to 93.4,\ \alpha\to\frac{1}{137},\ B_{11}\to 0.0594,\ B_{qq}\to 0.1350,\ B_{ggg}\to 0.877-0.1350\right\}\;/.$$
$$\left.\{\sigma\to 0.90\},\ \left\{s,\ (t-10)^2,\ (t+10)^2\right\}\right],\ \{t,\ 1546*2,\ 1551*2\}\Big]$$