Generalized Phase Space

Physics 230A, Spring 2007, Hitoshi Murayama

1 Symplectic Structure

In usual particle mechanics, the phase space is given by the coordinates q^i and their conjugate momenta p^i, and its volume by

$$V = \int \prod_{i=1}^{n} dp^i dq^i. \tag{1}$$

In some systems, however, the geometry of the phase space is not as simple. In general, we are talking about symplectic manifolds.

A symplectic manifold M is an $2n$-dimensional manifold that has a symplectic structure, namely that it admits a two-form $\omega = \frac{1}{2}\omega_{ij} dx^i \wedge dx^j$ called the symplectic form. The symplectic form has the following properties: (1) Closed, namely $d\omega = 0$, (2) Non-degenerate, namely the matrix $\omega_{ij} = -\omega_{ji}$ is invertible. Because of the second property, $\det\omega_{ij} \neq 0$, and hence $dV = \frac{1}{n!}\omega^n \neq 0$, which is the volume form of the phase space,

$$V = \int dV = \int \frac{1}{n!}\omega^n = \int (\det\omega_{ij}) \prod_{i=1}^{2n} dx^i. \tag{2}$$

Because the symplectic form is non-degenerate, we can define the inverse

$$\omega^{ij}\omega_{jk} = \delta^i_k. \tag{3}$$

For the simple case above, the symplectic form is

$$\omega = \sum_{i=1}^{n} (dp^i \wedge dq^i), \tag{4}$$

and the symplectic matrix is

$$\omega_{ij} = \left(\begin{array}{cc|cc|cc|cc} 0 & 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & -1 & 0 & \cdots & \cdots & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & -1 & 0 \end{array}\right). \tag{5}$$

One major restriction on symplectic manifolds, if compact, is that they need to have non-trivial second cohomology to allow for a closed non-degenerate two-form. Namely there is a two-dimensonal subsurface of the manifold that is closed (two-cycle C_2) on which the symplectic form can be integrated. This puts an interesting requirement on the normalization of the symplectic form as you will see in the next section. Kähler manifolds (complex manifolds of $U(N)$ holonomy) are all symplectic.

2 Hamiltonian

The Hamiltonian is a function on the phase space that generates the time translation. For the particle mechanics, we have the Hamilton equations of motion

$$\frac{dp^i}{dt} = -\frac{\partial H}{\partial q^i} \tag{6}$$

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p^i}. \tag{7}$$

The generalization of them is

$$\frac{dx^i}{dt} = \omega^{ij}\frac{\partial H}{\partial x^j} \tag{8}$$

Mathematicians like to talk about the *Hamiltonian vector*,

$$\frac{d}{dt} = \omega^{ij}\frac{\partial H}{\partial x^j}\frac{\partial}{\partial x^i}\ . \tag{9}$$

3 Poisson Bracket

The Poisson brackets are defined using the inverse of the symplectic matrix as

$$\{A, B\} = \omega^{ij} \frac{\partial A}{\partial x^i} \frac{\partial B}{\partial x^j} \ . \tag{10}$$

For the conventional case of p^i and q^i, it reduces to the standard definition

$$\{A, B\} = \frac{\partial A}{\partial p^i} \frac{\partial B}{\partial q^i} - \frac{\partial A}{\partial q^i} \frac{\partial B}{\partial p^i} \ . \tag{11}$$

Using this definition, it is easy to see that the Hamilton equations of motion can be rewritten as

$$\frac{dx^i}{dt} = \omega^{ij} \frac{\partial H}{\partial x^j} = \{x^i, H\}. \tag{12}$$

4 Path Integral

The Lagrangian on the phase space is normally

$$L = \sum_i p^i \dot{q}^i - H(p, q). \tag{13}$$

It is the first term that dictates the canonical structure. Note that the time integral of the first term is

$$\int \sum_i p^i \dot{q}^i dt = \int \sum_i p^i dq^i = \int \chi, \tag{14}$$

where the one-form $\chi = \sum_i p^i dq^i$ satisfies $\omega = d\chi$. This is true in general. Because the symplectic form is closed $d\omega = 0$, one can always write it locally exact $\omega = d\chi$.

In quantum mechanics, the action is exponentiated in the path integral,

$$\int \prod_t \prod_{i=1}^n (dq^i(t) dp^i(t)) e^{\frac{i}{\hbar} \int dt (\sum_i^n p^i \dot{q}^i - H(p,q))}. \tag{15}$$

The generalization of this expression is

$$\int \prod_t \left(\frac{1}{n!} \omega^n(t) \right) e^{\frac{i}{\hbar} (\int \chi - \int dt H)}. \tag{16}$$

For a closed path in the path integral, the integrand is

$$e^{i \oint_{C_1} \chi/\hbar} \tag{17}$$

Here, C_1 is the closed path in M. Using Stokes' theorem, it can be rewritten as

$$e^{i \int_D \omega/\hbar}, \tag{18}$$

where the disk D has the boundary $\partial D = C_1$. An interesting point is that, if M is compact, there are (at least) two disks D_1 and D_2 that cannot be continuously deformed from each other because of the non-trivial two-cycle C_2. The path integral must be well-defined independent of which disk is chosen. Note that the integral of ω does not change as the disk is continuously deformed because $d\omega = 0$. $D_1 \cup D_2$ can be deformed to C_2 and the integral must satisfy

$$e^{i \int_{C_2} \omega/\hbar} = 1 \tag{19}$$

so that the path integral does not depend on the choice of the disk. Therefore,

$$\int_{C_2} \omega = 2\pi n\hbar \qquad (n \in \mathbb{Z}). \tag{20}$$

This requirement is supposed to hold for any non-trivial two-cyles in M. This is analogous to the quantization condition of the monopole magnetic charge, see E. Witten, "Global Aspects of Current Algebra," *Nucl. Phys.* **B223**, 422-432 (1983).

5 Lagrangian

The Lagrangian submanifold is the generalization of the coordinate space $\{q^i\}$. The formal definition is that a submanifold $L \subset M$ is Lagrangian if it is an isotropic submanifold of dimension n, and the word isotropic means that the symplectic form vanishes on the tangent space TL, namely $\omega(v_1, v_2) = 0$ for $\forall v_i \in TL$. Simply put, ω combines coordinates and monenta, and if you pick only coordinates it vanishes.

We eliminate the rest of the coordinates from the phase-space Lagrangian and go back to the usual Lagrangian written in terms of the coordinates alone. In other words, we "integrate out" the canonical momenta from the path integral and arrive at the coordinate-space path integral with the Lagrangian which depends only on the canonical coordinates.

6 Example of Two-Sphere

The simplest but non-trivial example of a compact phase space is a two-dimensional sphere $S^2 = SU(2)/U(1)$. The symplectic form is nothing but the surface area of the sphere,

$$\omega = J \sin\theta d\phi \wedge d\theta. \tag{21}$$

It is trivially closed $d\omega = 0$ because there are no three-form on a two-dimensional space, and non-degenerate because the symplectic matrix

$$\omega_{ij} = \begin{pmatrix} 0 & J\sin\theta \\ -J\sin\theta & 0 \end{pmatrix} \tag{22}$$

is invertible. The Poisson bracket is

$$\{A, B\} = -\frac{1}{J\sin\theta}\left(\frac{\partial A}{\partial\phi}\frac{\partial B}{\partial\theta} - \frac{\partial A}{\partial\theta}\frac{\partial B}{\partial\phi}\right). \tag{23}$$

The symplectic form is locally exact

$$\omega = d\chi = d(J\cos\theta d\phi) \tag{24}$$

and hence one can write the Lagrangian

$$L = J\cos\theta\dot{\phi} - H(\theta, \phi). \tag{25}$$

The consistency of the path integral requires

$$\int_{S^2} \omega = 2\pi\hbar N, \tag{26}$$

and hence $J = j\hbar$ where $j = N/2$ is a half-integer.

The quantum mechanics of this phase space gives rise to the standard representation of $SU(2)$. For more details, consult "Models of spin," a miscellaneous note I wrote for 221A.

This is the simplest case of the general groups as briefly summarized in the next section.

7 General G/T

For a simple compact group G, and its maximal torus T (the abelian subgroup generated by its Cartan generators), the coset space G/T is Kähler, and hence symplectic. The quantum mechanism on the phase space G/T gives a Hilbert space of a representation of G, depending on what symplectic form is chosen. If the rank of the group G is r, $T \simeq U(1)^r$. Using the exact sequence of homotopy groups,

$$0 = \pi_2(G) \to \pi_2(G/T) \to \pi_1(T) = \mathbb{Z}^r \to \pi_1(G) = 0 \to \pi_1(G/T) \to \pi_0(T) = 0, \tag{27}$$

we find

$$\pi_2(G/T) = \mathbb{Z}^r, \qquad \pi_1(G/T) = 0. \tag{28}$$

Because π_1 vanishes, $H_2(G/T) = \pi_2(G/T) = \mathbb{Z}^r$, and hence there are r independent non-trivial two-cycles. On each of the two-cycle, we can associate an exact two-form which is quantized to obtain well-defined path integral. Thereore the symplectic form is a sum of unit two-forms with integer coefficients, which is equivalent to specifying the heightest weight of a representation as a sum of r fundamental weights with integer coefficients. The wave functions of the Hilbert space are given by the holomorphic sections of the complex line bundle whose first Chern character is the symplectic form. These wave functions form the representation of the group G. This is a consequence of the Borel–Bott–Weil theorem.

In general, the procedure called "geometric quantization" is to build a complex line bundle on M whose first Chern character is the symplectic form. Then the vector space of holomorphic sections is the Hilbert space of the quantum mechanics on the phase space M.