Regularization

Physics 230A, Spring 2007, Hitoshi Murayama

Introduction

In quantum field theories, we encounter many apparent divergences. Of course all physical quantities are finite, and therefore divergences appear only at intermediate stages of calculations that get cancelled one or the other way. However, such apparent divergences pose technical problems in dealing with them. Obviously we need some methods to add, subtract, multiply, and divide apparently divergent quantities and extract finite answers in the end. To do so, we need to "regulate" the divergences, namely to make apparent divergences manifestly finite so that we can manipulate them. This notes describe various ways of doing so using explicit examples.

What regularization does is to introduce a new parameter, let's say ϵ, to the apparently divergent quantity O. The quantity is now a function of ϵ, $O(\epsilon)$. It is supposed to reduce to the original quantity in the limit $\epsilon \to 0$

$$\lim_{\epsilon \to 0} O(\epsilon) = O, \tag{1}$$

you recover the apparent divergence. Yet for finite but very small ϵ, the quantity is finite $|O(\epsilon)| < \infty$. Then we say the divergent quantity O is *regularized* by the *regulator* ϵ.

One of the main issues of the regularization is that a regulator tends to break certain symmetries of the quantity. The usefulness of a regulator depends on what symmetries it retains, how easy it is to deal with, how widely it can be used, etc. In quantum field theories, we see cutoff regularization, Pauli–Villars regularization, dimensional regularization, ζ-function regularization, lattice regularization, etc. We look at a few explicit examples.

Example I

Consider the one-loop integral in ϕ^4 theory in Euclidean two dimensions,

$$\Sigma = \int \frac{d^2 p}{(2\pi)^2} \frac{1}{p^2 + m^2}. \tag{2}$$

This integral is logarithmically divergent at large momenta that we would like to regulate. Note that Σ is dimensionless.

(a) Sharp Cutoff

One way is to simply limit the momentum integral up to $p^2 \leq \Lambda^2$. We find

$$\begin{aligned}
\Sigma &= \int_0^{\Lambda^2} \frac{dp^2}{4\pi} \frac{1}{p^2+m^2} \\
&= \frac{1}{4\pi} \left[\ln(p^2+m^2)\right]_0^{\Lambda^2} \\
&= \frac{1}{4\pi} \ln \frac{\Lambda^2+m^2}{m^2}. \qquad (3)
\end{aligned}$$

In practice, this type of sharp brute-force cutoff is awkward and not convenient. One reason is that it explicitly breaks translational invariance in the momentum space $p \to p+k$ which you need to do when you combine several propagators into a single one using the Feynman parameters. Another reason is that it is difficult to maintain gauge invariance because $p_\mu = i\partial_\mu$ is not gauge covariant while $D_\mu = \partial_\mu - igA_\mu$ is. Nonetheless in theories which admit such a cutoff consistently, it has been used, *e.g.*, J. Polchinski, "Renormalization And Effective Lagrangians," Nucl. Phys. B **231**, 269 (1984).

(b) Gaussian Cutoff

Another but a similar way is the Gaussian cutoff

$$\Sigma = \int \frac{d^2p}{(2\pi)^2} \frac{e^{-p^2/\Lambda^2}}{p^2+m^2}. \qquad (4)$$

We find

$$\begin{aligned}
\Sigma &= \int \frac{d^2p}{(2\pi)^2} \int_{1/\Lambda^2}^{\infty} dt e^{-t(p^2+m^2)} e^{m^2/\Lambda^2} \\
&= \frac{e^{m^2/\Lambda^2}}{4\pi} \int_{1/\Lambda^2}^{\infty} \frac{dt}{t} e^{-tm^2} \\
&= \frac{e^{m^2/\Lambda^2}}{4\pi} \int_{m^2/\Lambda^2}^{\infty} \frac{dt'}{t'} e^{-t'} \\
&= -\frac{e^{m^2/\Lambda^2}}{4\pi} \mathrm{Ei}(-m^2/\Lambda^2). \qquad (5)
\end{aligned}$$

Here, $\mathrm{Ei}(x)$ is the exponential integral function. It has Taylor expansion for large Λ,

$$\Sigma = \frac{1}{4\pi}\left[\left(\log\frac{\Lambda^2}{m^2} - \gamma\right) + \frac{m^2}{\Lambda^2}\left(2\log\frac{\Lambda^2}{m^2} + 1 + \gamma\right) + O\left(\frac{m^2}{\Lambda^2}\right)^2\right]. \quad (6)$$

This regularization can be implemented by the modified kinetic term in the Lagrangian

$$\frac{1}{2}\partial_\mu\phi e^{\Box/\Lambda^2}\partial_\mu\phi, \quad (7)$$

which modifies each propagator to $e^{-p^2/\Lambda^2}/(p^2+m^2)$. It still suffers from the lack of translational invariance in the momentum space, but it can in principle be made gauge-invariant by replacing $\Box = \partial_\mu\partial_\mu$ by $D_\mu D_\mu$.

(c) Higher Derivative Regularization

Yet more related regularization is called higher derivative regularization. It also modifies the quadratic terms such as

$$\frac{1}{2}\partial_\mu\phi\left(1 - \frac{\Box}{\Lambda^2}\right)\partial_\mu\phi + \frac{1}{2}m^2\phi\left(1 - \frac{\Box}{\Lambda^2}\right)\phi. \quad (8)$$

Then the propagator is modified to

$$\left(1 + \frac{p^2}{\Lambda^2}\right)^{-1}\frac{1}{p^2+m^2} = \frac{\Lambda^2}{(p^2+\Lambda^2)(p^2+m^2)}. \quad (9)$$

The one-loop diagram is

$$\begin{aligned}
\Sigma &= \int\frac{d^2p}{(2\pi)^2}\frac{\Lambda^2}{(p^2+\Lambda^2)(p^2+m^2)} \\
&= \int_0^1 dz\frac{d^2p}{(2\pi)^2}\frac{\Lambda^2}{(p^2+z\Lambda^2+(1-z)m^2)^2} \\
&= \int_0^1 dz\frac{1}{4\pi}\frac{\Lambda^2}{z\Lambda^2+(1-z)m^2} \\
&= \frac{1}{4\pi}\frac{\Lambda^2}{\Lambda^2-m^2}\ln\frac{\Lambda^2}{m^2}. \quad (10)
\end{aligned}$$

This one also still suffers from the lack of translational invariance in the momentum space, but it can again in principle be made gauge-invariant by replacing $\Box = \partial_\mu\partial_\mu$ by $D_\mu D_\mu$.

(d) Pauli–Villars Regularization

Pauli–Villars regularization subtracts off the same loop integral with a much larger mass,

$$\Sigma = \int \frac{d^2p}{(2\pi)^2} \left(\frac{1}{p^2+m^2} - \frac{1}{p^2+M^2} \right) = \frac{1}{4\pi} \ln \frac{M^2}{m^2} \,. \tag{11}$$

This method has the benefit of maintaining the translational invariance in the momentum space. It also maintains the gauge invariance that can be seen the following way. The subtracted piece is regarded as a contribution of another field (Pauli–Villars field) with the same quantum numbers as the original field, but has the opposite statistics. For instance, in ϕ^4 theory, the Pauli–Villars field Φ is fermion and is Grassmann-odd number in path integral even though it has the same quantum number and hence is a scalar field. Of course a fermionic scalar field would break spin-statistics theorem and leads to violation of causality and/or positivity of energy. However, as long as we take its mass M very large and deal with physics at energy scales much lower than M, such diseases do not appear. Unfortunately, it is still not appropriate in chiral gauge theories because they do not allow mass terms for fermions.[1]

(e) Dimensional Regularization

Dimensional regularization assumes that the spacetime dimension is not two but is analytically continued to $2-2\epsilon$

$$\Sigma = \mu^{2\epsilon} \int \frac{d^{2-2\epsilon}p}{(2\pi)^{2-2\epsilon}} \frac{1}{p^2+m^2}. \tag{12}$$

Here, μ is an arbitrary parameter of dimension of energy to force the result Σ to be dimensionless. The result is

$$\begin{aligned} \Sigma &= \mu^{2\epsilon} \frac{1}{(4\pi)^{1-\epsilon}} \Gamma(\epsilon) (m^2)^{-\epsilon} \\ &= \frac{1}{(4\pi)^{1-\epsilon}} \left[\frac{1}{\epsilon} - \gamma + \frac{1}{2} \left(\gamma^2 + \frac{\pi^2}{6} \right) \epsilon + O(\epsilon^2) \right] \left(\frac{m^2}{\mu^2} \right)^{-\epsilon} \end{aligned}$$

[1] One way around this problem is to introduce an *infinite* number of Pauli–Villars fields with mass $M_n = nM_1$ with alternating statistics. See S. A. Frolov and A. A. Slavnov, "An Invariant Regularization Of The Standard Model," Phys. Lett. B **309**, 344 (1993).

$$= \frac{1}{4\pi}\left[\frac{1}{\epsilon} - \gamma + \ln 4\pi - \ln\frac{m^2}{\mu^2} + O(\epsilon)\right]. \tag{13}$$

Compared to the result of the sharp cutoff, we can identify the effective cutoff to be $\Lambda^2 = 4\pi\mu^2 e^{1/\epsilon - \gamma}$. This regularization has many advantages: translational invariance in the momentum space, gauge invariance. The drawback is obvious: it is highly artificial to discuss spacetime in non-integer dimensions. Also, it is tricky to extend the Dirac gamma matrices to non-integer dimensions. In general, the number of degrees of freedom changes from two to $2-2\epsilon$ dimensions which cause problems.

(f) Dimensional Reduction Regularization

This is technically a small modification of the dimensional regularization, yet it is much better conceptually as the spacetime dimension is maintained at two. It is assumed that 2ϵ dimensions out of two are "compactified" into small size so that the momenta along these 2ϵ directions are quantized and we consider only the zero modes. Then the momentum integrals are done only in $2-2\epsilon$ dimensions. However the Dirac gamma matrices remain the same as in two dimensions. The vector fields now decompose into the vector fields in the remaining $2-2\epsilon$ dimensions and the components in the 2ϵ dimensions that have lost the indices and appear as scalars. The latter are called "ϵ-scalars". This way, the momentum integrals are regulated yet the number of degrees of freedom does not change.

For instance, if we compute the closed loop of fermions with the Yukawa coupling $\bar{\psi}\psi\phi$ for the tadpole diagram of the scalar ϕ, it is

$$-\int \frac{d^{2-2\epsilon}p}{(2\pi)^2}\frac{\mathrm{Tr}(\not{p}+m)}{p^2+m^2}. \tag{14}$$

When taking the trace over the Dirac indices, it is taken as if it is still in two dimensions, and hence $\mathrm{Tr}(\not{p}+m) = 2m$. Similarly, a loop of gauge boson would be $g_{\mu\nu}g^{\mu\nu} = 2-2\epsilon$, and the contribution of ϵ-scalars is added with 2ϵ of them. The total reproduce the original 2 components of the vector field. Note, however, that the ϵ-scalars may receive separate counter terms from the remaining $2-2\epsilon$ vector boson because they are independent fields now.

Example II

Consider the following one-parameter integral,

$$F(a) = \int_0^\infty \frac{dt}{t} e^{-at}. \tag{15}$$

This integral is divergent at the boundary $t = 0$. If one deals with it incorrectly, one may conclude that it does not depend on the parameter a at all, since one can make a change of the variable $t = t'/a$ as

$$F(a) = -\int_0^\infty \frac{dt'}{t'} e^{-t'} \tag{16}$$

where the parameter a has completely dropped out. Namely this integral is apparently *scale-invariant*. However, this invariance cannot be retained by any of the regularizations below, a common phenomenon in quantum field theories. In the following, imagine that the parameter a has a dimension of energy squared, and t the inverse energy squared. The quantity $F(a)$ is dimensionless.

The simplest way to deal with this divergence is to cutoff the integral at $t = 1/\Lambda^2$,

$$F_\Lambda(a) = -\int_{1/\Lambda^2}^\infty \frac{dt}{t} e^{-at}. \tag{17}$$

By the same change of variable, we find

$$F_\Lambda(a) = -\int_{a/\Lambda^2}^\infty \frac{dt'}{t'} e^{-t'} = \mathrm{Ei}(-a/\Lambda^2) \tag{18}$$

This is so-caled exponential integral function $-\mathrm{Ei}(-a/\Lambda^2)$. It has Taylor expansion for large Λ,

$$F_\Lambda(a) = \log \frac{a}{\Lambda^2} + \gamma - \frac{a}{\Lambda^2} + \frac{a^2}{4\Lambda^4} + O(\Lambda^{-3}). \tag{19}$$

It is indeed divergent as $\Lambda \to \infty$, but the leading dependence on a is clear: $\log a$. The cutoff Λ has the dimension of energy, namely a high-energy cutoff.

Pauli–Villars regularization takes a difference of a divergent quantity with a similar quantity with a different parameter. Namely,

$$F(a) - F(A) = -\int_0^\infty \frac{dt}{t}(e^{-at} - e^{-At}). \tag{20}$$

Because the new term in the integrand vanishes in the limit $A \to \infty$, we expect that the original quantity is recovered in this limit. This integral converges and is simply

$$F(a) - F(A) = \log \frac{a}{A}. \tag{21}$$

Again the result is divergent when the regulator is removed $A \to \infty$, while we find the same leading dependence on a: $\log a$. However, the finite terms are different from the cutoff regularization, and one needs to consistently use the same regularization until the manifestly finite result is obtained. The physical meaning of A is similar to the cutoff in energy Λ above, yet the way energy is cutoff here is more smooth. Their naive comparison says $A = \Lambda^2 e^{-\gamma}$.

The dimensional regularization in this example is

$$F_\epsilon(a) = -\mu^{2\epsilon} \int_0^\infty \frac{dt}{t^{1-\epsilon}} e^{-at} = -\mu^{2\epsilon} a^{-\epsilon} \int_0^\infty dt' \; t'^{\epsilon-1} e^{-t'} = -\Gamma(\epsilon) \left(\frac{a}{\mu^2} \right)^{-\epsilon}. \tag{22}$$

Note that we had to introduce an arbitrary energy scale μ to keep the quantity dimensionless. The expression has a Taylor expansion in the regulator,

$$F_\epsilon(a) = -\left[\frac{1}{\epsilon} - \gamma + \frac{1}{2} \left(\gamma^2 + \frac{\pi^2}{6} \right) \epsilon + O(\epsilon^2) \right] \left(\frac{a}{\mu^2} \right)^{-\epsilon} = -\frac{1}{\epsilon} + \gamma + \log \frac{a}{\mu^2} + O(\epsilon). \tag{23}$$

It has the same dependence on a as other regulators, while the expression is again different. A naive comparison with the cutoff regularization says $\Lambda^2 = \mu^2 e^{1/\epsilon}$.