Field Theory Techniques on Spin Systems
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1 Introduction

Solid state systems are based on crystals that have discretized positions for electronic degrees of freedom. In other words, the atomic distance provides a short-distance cutoff to the system below which you have to change the description of the system to that of nuclei and electrons instead of atoms. The intrinsic inability of the atomic description below the Bohr radius is not a problem, as long as we are aware of its limitation. Any physical theories have their limited applicability we should be aware of and we can live with.

Despite the intrinsic distance scale in the problem, the atomic scale, many solid state systems exhibit interesting phenomena at distances much larger than atoms. This is particularly true when the system is close to a second-order phase transition. Imagine a ferromagnet described by the Heisenberg model,

$$H = -\frac{J}{\hbar^2} \sum_{\langle i,j \rangle} \vec{s}_i \cdot \vec{s}_j. \tag{1}$$

Here and below, the sum over $\langle i, j \rangle$ refers to nearest neighbors on the lattice. The parameter J has the dimension of energy, describing the strength of the spin-spin interaction. The origin of the interaction is actually the Coulomb replusion between electrons which prefers the orbital wave function to be anti-symmetric than symmetric to separate their positions, requiring symmetric spin wave function to be consistent with the overall anti-symmetry of fermion wave functions. Therefore J is expected to be of the order of electronic energies, *i.e.*, electronvolt.

At a high temperature, all spins are random due to their thermal fluctuations. As one lowers the temperature, one reaches the critical temperature at which all spins start to line up. How do spins know which direction to point? Somehow each spin becomes very susceptible to all other spins over a macroscopic (say, many microns) distance so that it knows which way to point?

It is like spins become very "fashion-conscious" close to the critical temperature. Imagine being in a densely-packed dance hall where you can only see people around you. But you are very keen to know what other groups in the same hall are doing. When they start a new fashion or trend, you'd like to join the trend right away. Due to some reason spins manage to do so. When a group of spins start to line up, other spins far away can tell and start lining up with them, too.

This long-range correlation used to be a big mystery. What we now know is that the correlation length diverges at the critical temperature of a second-order phase transition so that each spin "learns" what the others are doing.

Mathematically, the long-range correlation is expressed in terms of the correlation function

$$\langle s^i(0)s^j(\vec{x})\rangle \propto \delta^{ij}e^{-r/\xi}. \tag{2}$$

For a finite ξ, the correlation function damps exponentially at large distances and a spin cannot tell what a far-away spin is doing. But as the temperature approaches the critical point $T \to T_c$, the correlation length diverges $\xi \to \infty$. This is the phenomenon we would like to understand using the quantum field theory techniques.

In order to use techniques in continuum field theory to study systems on a lattice, we first need to derive the continuum Lagrangian from the given lattice system.

2 Heisenberg Model

The Heisenberg model of magets (ferromagnets) is given by the Hamiltonian

$$H = -\frac{J}{\hbar^2}\sum_{\langle i,j\rangle}\vec{s}_i\cdot\vec{s}_j. \tag{3}$$

The sum over $\langle i,j\rangle$ refers to the nearest neighbor sites on the lattice. What we would like to do now is to rewrite this Hamiltonian in terms of continuum field theory.

The first step is to use the path integral formulation of spins, as described in a separate lecture notes. Each spin is described by a vector $\vec{s} = j\hbar\vec{n} = j\hbar(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ with the action

$$S = j\hbar\int\cos\theta\dot{\phi}dt. \tag{4}$$

One can also use the complex coordinate $z(t)$ which is related to the unit vector $\vec{n}$ by

$$n_z = \frac{1-\bar{z}z}{1+\bar{z}z}, \qquad n_x + i n_y = \frac{2z}{1+\bar{z}z}. \tag{5}$$

It is easy to verify that $\vec{n}^2 = 1$ using this definition. This complex coordinate is the stereographic projection of the point on the sphere from the north pole down to the plane tangent to the sphere at the south pole.

To put the Heisenberg model on a path integral, the first step is to use this Lagrangian for each spin. Then the action is

$$L = j\hbar \sum_i \cos\theta_i \dot{\phi}_i dt + \frac{J}{\hbar^2} \sum_{\langle i,j \rangle} (j\hbar)^2 \vec{n}_i \cdot \vec{n}_j. \tag{6}$$

Now we use the following trick:

$$\vec{n}_i \cdot \vec{n}_j = \frac{1}{2}[\vec{n}_i^2 + \vec{n}_j^2 - (\vec{n}_i - \vec{n}_j)^2] = 1 - \frac{1}{2}(\vec{n}_i - \vec{n}_j)^2. \tag{7}$$

We omit the constant term from the Lagrangian, and we now have

$$L = j\hbar \sum_i \cos\theta_i \dot{\phi}_i dt - \frac{1}{2} J j^2 \sum_{\langle i,j \rangle} (\vec{n}_i - \vec{n}_j)^2. \tag{8}$$

Assuming that the neighboring spins are more-or-less pointing the same direction due to this interaction, we can regard the unit vectors as a continuous field $\vec{n}(\vec{x}, t)$, and the difference between neighboring sites is approximated as

$$\vec{n}_i - \vec{n}_j = (\vec{a} \cdot \vec{\nabla})\vec{n} + O(a)^2, \tag{9}$$

where $\vec{a}$ is one of the lattice vectors. Let us assume a cubic lattice of lattice constant a, so that it is $a\nabla_{x,y,z}\vec{n}$. The sum over nearest neighbors then becomes,

$$\begin{aligned} \sum_{\langle i,j \rangle} (\vec{n}_i - \vec{n}_j)^2 &= \int \frac{d\vec{x}}{a^D} a^2 \left[(\nabla_x \vec{n})^2 + (\nabla_y \vec{n})^2 + (\nabla_z \vec{n})^2 + O(a)^2 \right] \\ &= \frac{1}{a^{D-2}} \int d\vec{x} (\vec{\nabla} n^i)^2. \end{aligned} \tag{10}$$

Here, the repeated index i is summed over the three components of the spin $i = 1, 2, 3$. Finally, the sum over the sites for the first term in the Lagrangian is also approximated by the continuum field and we finally find

$$L = \int d\vec{x} \left[\frac{1}{a^D} j\hbar \cos\theta \dot{\phi} - \frac{1}{2}\frac{1}{a^{D-2}} J j^2 (\vec{\nabla} n^i)^2 \right]. \tag{11}$$

In order to derive the equation of motion, we must vary n^i subject to the constraint $n^i n^i = 1$. This is most easily done using the complex coordinate z. Using the definition Eq. (5), it is easy to see that

$$L = \int d\vec{x} \left[\frac{1}{a^D} j\hbar \frac{2\bar{z} i \dot{z}}{1+\bar{z}z} - \frac{1}{2}\frac{1}{a^{D-2}} J j^2 \frac{4\vec{\nabla}\bar{z}\cdot\vec{\nabla}z}{(1+\bar{z}z)^2} \right]. \tag{12}$$

The Euler–Lagrange equation of motion is

$$\frac{1}{a^D} j\hbar \frac{2i\dot{z}}{(1+\bar{z}z)^2} + \frac{1}{2}\frac{1}{a^{D-2}} J j^2 \left[\frac{4\vec{\nabla}^2 z}{(1+\bar{z}z)^2} + \frac{4\vec{\nabla}\bar{z}\cdot\vec{\nabla}z}{(1+\bar{z}z)^3} z \right] = 0. \tag{13}$$

Clearly, constant z and $\bar{z}$ is a static solution. It is nothing but the state where all the spins are lined up along the same direction.

The excitation above the ground state can be studied by "linearing" the equation, namely that we expand the equation of motion up to the first order in the fluctuation. Or equivalently, we expand the Lagrangian up to the quadratic order in fluctuations. The simplest choice of the ground state for us with this coordinate is obviously $z = \bar{z} = 0$. Then z and $\bar{z}$ are fluctuations. The Lagrangian expanded up to the quadratic order is

$$L = \int d\vec{x} \left[\frac{1}{a^D} j\hbar 2\bar{z} i \dot{z} - \frac{1}{2}\frac{1}{a^{D-2}} J j^2 4\vec{\nabla}\bar{z}\cdot\vec{\nabla}z + O(z,\bar{z})^4 \right]. \tag{14}$$

The equation of motion then is

$$\frac{1}{a^D} j\hbar 2i\dot{z} + \frac{1}{2}\frac{1}{a^{D-2}} J j^2 4\vec{\nabla}^2 z = 0. \tag{15}$$

Going to the Fourier space $z = z_n e^{-i(Et-\vec{p}\cdot\vec{x})/\hbar}$, we find

$$\frac{1}{a^D} j\hbar 2\frac{E}{\hbar} - \frac{1}{2}\frac{1}{a^{D-2}} J j^2 4\frac{\vec{p}^2}{\hbar^2} z = 0, \tag{16}$$

and hence

$$E(\vec{p}) = J j \frac{\vec{p}^2 a^2}{\hbar^2}. \tag{17}$$

This agrees completely with the excitation spectrum worked out using the operators in Lecutre notes on Spontaneous Symmetry Breaking,

$$E(k) = J(1 - \cos ka) \tag{18}$$

by identifying $k = |\vec{p}|/\hbar$, $j = 1/2$, and expanding the expression up to the second order in k (because we kept only second order in derivatives).

Now we move to the finite temperature. The partition function is obtained by the path integral for the imaginary time $t = -i\tau$ with the periodic boundary condition $z(\tau + \hbar\beta) = z(\tau)$ etc for the bosonic fields. (For fermion fields, anti-periodic boundary condition is used instead. See Lecture notes on path integral.) The imaginary-time action is

$$S = \oint_0^{\hbar\beta} d\tau \int d\vec{x} \left[\frac{1}{a^D} j\hbar \frac{2\bar{z}i\dot{z}}{1+\bar{z}z} + \frac{1}{2}\frac{1}{a^{D-2}} J j^2 \frac{4\vec{\nabla}\bar{z}\cdot\vec{\nabla}z}{(1+\bar{z}z)^2} \right]. \tag{19}$$

We can expand the fields in the Fourier modes,

$$z(\vec{x},\tau) = \sum_{n=-\infty}^{\infty} z_n(\vec{x}) e^{-in\tau/\hbar\beta}. \tag{20}$$

Then the linearized action is

$$S = \hbar\beta \sum_n \int d\vec{x} \left[\frac{1}{a^D} j\hbar 2\bar{z}_n \frac{n}{\hbar\beta} z_n + \frac{1}{2}\frac{1}{a^{D-2}} J j^2 4\vec{\nabla}\bar{z}_n \cdot \vec{\nabla} z_n + O(z,\bar{z})^4 \right]. \tag{21}$$

Therefore, the correlation function of z_n is damped exponentially as e^{-r/ξ_n} with the correlation length

$$\xi_n^2 = j\frac{\beta J}{n} a^2. \tag{22}$$

Remember J is of the order of electronvolt, the Curie (phase transition) temperature typically $T_c \sim 1000$K. Therefore $\beta J \sim 10$, and hence the correlation length is not much larger than the lattice constant. They do not give rise to long-range correlation we expect at the second-order phase transition. We can safely integrate them out from the theory, keeping only the zero mode z_0. The path integral over the non-zero modes may change the action, but the $SO(3)$ symmetry of the system guarantees that the only corrections one may obtain are of the same form as the terms we already have (up to higher order derivatives which we can ignore to study long-range correlations), namely just the renormalization of the coefficients. In other words, at T_c and long distances, the periodic imaginary time direction is so much flatted out that we can ignore it, and retain only the spatial dimensions.

Here, we simply set $z_n = 0$ and keep the zero mode,

$$S = \hbar\beta \int d\vec{x} \left[\frac{1}{2} \frac{1}{a^{D-2}} J j^2 \frac{4\vec{\nabla}\bar{z} \cdot \vec{\nabla} z}{(1+\bar{z}z)^2} \right] = \hbar\beta \int d\vec{x} \left[\frac{1}{2} \frac{1}{a^{D-2}} J j^2 (\vec{\nabla} n^i)^2 \right]. \quad (23)$$

Writing the sum over the index i explicitly, the partition function is

$$Z = \int \prod_{i=1}^{3} \mathcal{D}n^i(\vec{x}) e^{-\frac{1}{2g_0^2} \int d\vec{x} \sum_i^3 (\vec{\nabla} n^i)^2} \delta(\sum_{i=1}^{3} n^i n^i - 1), \quad (24)$$

where

$$\frac{1}{g_0^2} = \beta J \frac{j^2}{a^{D-2}}. \quad (25)$$

g_0 has length dimension $(D-2)/2$. In particular, it is dimensionless in $D = 2$. It is useful to remember that the bare coupling squared g_0^2 is proportional to the temperature.

There is interest in other values of N as well. $N = 1$ corresponds to the Ising model. $N = 2$ is the so-called XY model, which describes the long-range behavior of superfluid.

3 Non-Linear Sigma Model

Unfortunately we cannot solve the field theory exactly for most dimensionalities and coupling g_0. We will discuss later that perturbation theory is good around $D = 2$ or 4 using different forms of the Lagrangian. On the other hand we would like to understand general issues of spontaneous symmetry breaking for all dimensionalities. For this purpose, it is useful to study the model

$$Z = \int \prod_{i=1}^{N} \mathcal{D}n^i(\vec{x}) e^{-\frac{1}{2g_0^2} \int d\vec{x} (\vec{\nabla} n^i)^2} \delta(\sum_{i=1}^{N} n^i n^i - 1), \quad (26)$$

for general N. It turns out that this model is exactly solvable for the limit $N \to \infty$. This model is called $SO(N)/SO(N-1)$ Non-Linear Sigma Model.

The Heisenberg model corresponds to $N = 3$, where n^i sweeps the surface of two-sphere S^2. For $N = 1$, $n = \pm 1$, and the model describes the Ising model. For $N=2$, n^i sweeps a circle S^1, and the model is called XY model, which appears in the study of superfluids and hexatic liquid crystals. $N>3$ is not used to describe any realistic condensed matter systems as far as I know. Nonetheless, it is useful to keep N as a free parameter, and in particular we will consider the limit $N \to \infty$ which gives us the exact solution to the system. n^i sweeps the surface of $N-1$-dimensional sphere S^{N-1}.

As explained in Peskin–Schroeder, we rewrite the Lagrangian as

$$Z = \int \prod_{i=1}^{N} \mathcal{D}n^i(\vec{x})\mathcal{D}\alpha(\vec{x}) e^{-\frac{1}{2g_0^2}\int d\vec{x}\left[(\vec{\nabla}n^i)^2 + i\alpha(n^i n^i - 1)\right]}. \tag{27}$$

Here, the delta function that enforces $n^i n^i = 1$ is rewritten in terms of an integral over a (Lagrange-multiplier) field $\alpha(\vec{x})$. The coefficient $1/2g_0^2$ in front of α is there for the later convenience. Because the overall normalization of the partition function drops out from any physical correlation functions, we are free to use this coefficient.

The minimum energy field configuration is given by $n^i(\vec{x}) = n^i$ (constant), again consistent with all spins aligned, namely we expect an ordered state. However, the thermal fluctuations may destroy the order. This is one of the principal questions we would like to study close to the critical point.

If the spins are lined up and hence there is a long-range order, we expect $n^i \neq 0$ and hence the correlation function approaches a constant at large distances,

$$\langle n^i(0) n^j(\vec{x})\rangle \to \delta^{ij}|\langle n^i\rangle|^2 \neq 0 \qquad (|\vec{x}| \to \infty). \tag{28}$$

At the critical temperature, the order is just about to set in and hence the correlation function approaches zero. However, because there is a long-range correlation of "fashion-consciousness," it does not damp exponentially. It is supposed to follow a power-law damping,

$$\langle n^i(0) n^j(\vec{x})\rangle \simeq |\vec{x}|^{-(D-2+\eta)} \qquad (|\vec{x}| \to \infty), \tag{29}$$

This parameter η is one of the quantities called "critical exponents," describing the characteristic behavior of physical quantities close to the critical point.

If the system is disordered, the correlation function damps exponentially to zero,

$$\langle n^i(0) n^j(\vec{x})\rangle \simeq \delta^{ij} e^{-r/\xi} \qquad (|\vec{x}| \to \infty). \tag{30}$$

The parameter ξ is the correlation length. How quickly it diverges close to the critical temperature

$$\xi \sim \left(\frac{T - T_c}{T_c} \right)^{-\nu} \tag{31}$$

defines another critical exponent ν.

If one approaches the critical temperature from below, the magnetization goes to zero as

$$M \sim \left(\frac{T_c - T}{T_c} \right)^{\beta}. \tag{32}$$

The critical exponent β is also of interest.

3.1 Large N Exact Solutions

It is instructive to study the model first in the large N limit where an exact solution can be obtained. We will come back to the perturbation theory to see how it works for general N. This is the same discussion as in Section 13.3 of Peskin–Schroeder, but it skips some of the calculations and I present here more details.

For spatial dimensions larger than two, the coupling g_0 has length dimension $(D-2)/2$ and hence the theory is non-renormalizable. Since the theory has a natural cutoff at the atomic distance, this is not a problem.

The technique we use is to integrate over n^i and study the effective action for α. To do so, we have to make two assumptions. One is that α acquires a finite expectation value, so that n^i is massive and can be integrated out. Then we minimize the result with respect to α. The other is that α is spatially constant. Both of them can be justified *a posteori*, by calculating the effective action and show that α is non-vanishing at the minimum of the potential, and the energy increases for spatially-varying configuration. If this turns out to be the case, n^i is massive and hence its correlation function damps exponentially at large distances, signaling lack of long-range order. On the other hand, if α does not have an expectation value, it was not correct to integrate out n^i without assigning a vacuum expectation value. Then n^i stays massless and there is a long-range order. We will come back to this question shortly.

Under the assumption that there is a constant expectation value for α, we compute the effective potential,

$$
\begin{aligned}
Z &= \int \mathcal{D}\alpha(\vec{x}) e^{-\int d^D x V_{\mathit{eff}}(\alpha)}, \\
e^{-\int d^D x V_{\mathit{eff}}(\alpha)} &= \int \prod_{i=1}^{N} \mathcal{D}n^i(\vec{x}) e^{-\frac{1}{2g_0^2}\int d^D x \left[(\vec{\nabla} n^i)^2 + i\alpha(n^i n^i - 1)\right]} \\
&= [\det(-\vec{\nabla}^2 + i\alpha)]^{-N/2} e^{\int d^D x \frac{1}{2g_0^2} i\alpha}.
\end{aligned} \tag{33}
$$

Therefore, we find

$$
\int d^D x V_{\mathit{eff}}(\alpha) = \frac{N}{2}\left[\operatorname{Tr}\ln(-\vec{\nabla}^2 + i\alpha) - \int d^D x \frac{1}{N g_0^2} i\alpha\right]. \tag{34}
$$

We consider the limit where $N \to \infty$, while keeping $N g_0^2$ fixed. The latter is called 't Hooft coupling. We will see later that fixing this combination makes sense from the point of view of the perturbation theory. Then, the exponent changes so rapidly as a function of α that the steepest descent method can be used, namely that the integral is dominated by the stationary point of the effective potential. It is in this limit that we have the exact result: all we need to do is to solve $dV_{\mathit{eff}}(\alpha)/d\alpha = 0$.

Now we calculate the determinant.

$$
\begin{aligned}
\int d^D x V_{\mathit{eff}}(\alpha) &= \frac{N}{2}\left[\operatorname{Tr}\ln(-\vec{\nabla}^2 + i\alpha) - \int d^D x \frac{1}{N g_0^2} i\alpha\right] \\
&= \frac{N}{2}\left[\int \frac{d^D x d^D p}{(2\pi)^D} \ln(\vec{p}^2 + i\alpha) - \int d^D x \frac{1}{N g_0^2} i\alpha\right],
\end{aligned} \tag{35}
$$

and hence the superficial degree of divergence is D. However, we only need to solve the stationary condition, and hence we take the derivative with respect to α,

$$
\int \frac{d^D p}{(2\pi)^D} \frac{1}{\vec{p}^2 + i\alpha} - \frac{1}{N g_0^2} = 0. \tag{36}
$$

In order to work it out with a physical cutoff $p \sim 1/a$, where a is the lattice constant, we employ a Gaussian cutoff to smoothly remove the contributions from modes with high momenta. Because at the minimum we will find $i\alpha = m^2 > 0$, we use this notation anticipating the result. Therefore, we look for the solution to the equation

$$
\int \frac{d^D p}{(2\pi)^D} \frac{e^{-\vec{p}^2/\Lambda^2}}{\vec{p}^2 + m^2} = \frac{1}{N g_0^2}. \tag{37}
$$

This equation is sometimes called *gap equation* because it determines the "gap" which is the non-zero energy required to create the excitation n^i.

The momentum integral can be evaluated as

$$\begin{aligned}
\int \frac{d^D p}{(2\pi)^D} \frac{e^{-\vec{p}^2/\Lambda^2}}{\vec{p}^2 + m^2} &= \int \frac{d^D p}{(2\pi)^D} \int_{1/\Lambda^2}^{\infty} dt e^{-t(\vec{p}^2+m^2)} e^{m^2/\Lambda^2} \\
= \; e^{m^2/\Lambda^2} \frac{1}{(2\pi)^D} \int_{1/\Lambda^2}^{\infty} dt \left(\frac{\pi}{t}\right)^{D/2} e^{-tm^2} \\
= \; e^{m^2/\Lambda^2} (\Lambda^2)^{(D-2)/2} \frac{1}{(4\pi)^{D/2}} \int_1^{\infty} dt \; t^{-D/2} e^{-tm^2/\Lambda^2} \\
= \; e^{m^2/\Lambda^2} \frac{\Lambda^{D-2}}{(4\pi)^{D/2}} E_{D/2}(m^2/\Lambda^2).
\end{aligned} \tag{38}$$

Here, $E_n(z)$ is the exponential integral function

$$E_n(z) = \int_1^{\infty} \frac{e^{-zt}}{t^n} dt. \tag{39}$$

Note that the prefactor e^{m^2/Λ^2} is always smaller than the factor in the integrand e^{-tm^2/Λ^2} because $t > 1$, and hence the integral is bounded from above,

$$\begin{aligned}
\int \frac{d^D p}{(2\pi)^D} \frac{e^{-\vec{p}^2/\Lambda^2}}{\vec{p}^2 + m^2} &= \frac{\Lambda^{D-2}}{(4\pi)^{D/2}} \int_1^{\infty} dt \; t^{-D/2} e^{-tm^2/\Lambda^2} e^{m^2/\Lambda^2} \\
&< \frac{\Lambda^{D-2}}{(4\pi)^{D/2}} \int_1^{\infty} dt \; t^{-D/2} = \frac{\Lambda^{D-2}}{(4\pi)^{D/2}} \frac{1}{-1 + D/2}.
\end{aligned} \tag{40}$$

Note that the last integral does not converge for $D \leq 2$ and this upper bound applies only for $D > 2$. The existence of the upper bound is very important in the following discussions of the critical point. Namely that the equation Eq. (37) only if the coupling is larger than the critical coupling

$$N g_0^2 > N g_c^2 = \frac{(4\pi)^{D/2}}{\Lambda^{D-2}} \left(\frac{D}{2} - 1\right) \tag{41}$$

for $D > 2$. On the other hand, there is a solution for any coupling for $D = 2$ as we will see below.

For the purpose of studying each dimensions more explicitly, the integral can be expanded in power series in m^2/Λ^2. The stationary condition is

$$\frac{1}{128\pi^3} \left[\Lambda^4 - \Lambda^2 m^2 + \left(-\gamma - \ln \frac{m^2}{\Lambda^2}\right) m^4\right] = \frac{1}{N g_0^2} \tag{42}$$

for $D=6$,

$$\frac{1}{16\pi^2}\left[\Lambda^2+\left(\gamma+2\ln\frac{m^2}{\Lambda^2}\right)m^2\right]=\frac{1}{Ng_0^2} \tag{43}$$

for $D=4$,

$$-\frac{1}{4\pi}\left(\gamma+\ln\frac{m^2}{\Lambda^2}\right)=\frac{1}{Ng_0^2} \tag{44}$$

for $D=2$. For non-integer $D/2$, we can use the analytic continuation of the exponential integral function

$$\begin{aligned}E_{D/2}(\epsilon) &= \epsilon^{-1+D/2}\Gamma\left(1-\frac{D}{2}\right)\\ &+\left(-\frac{1}{1-D/2}+\frac{\epsilon}{2-D/2}-\frac{\epsilon^2}{2(3-D/2)}+\frac{\epsilon^3}{6(4-D/2)}-\cdots\right).\end{aligned} \tag{45}$$

On the other hand when $\alpha=0$ and $\langle n^i\rangle\neq 0$, we need to expand the Lagrangian around the solution $n^i=n^i_{cl}+\Delta n^i$, and integrate over Δn^i. See Section 11.4 of Peskin–Schroeder why this procedure is equivalent to add the source term, integrate over fields, do the inverse Legendre transform, and get the 1PI effective action. The effective potential Eq. (34) is changed to

$$\int d^Dx V_{\mathit{eff}}(\alpha,n^i_{cl})=\frac{N}{2}\left[\mathrm{Tr}\ln(-\vec{\nabla}^2+i\alpha)-\int d^Dx\frac{1}{Ng_0^2}(1-n^i_{cl}n^i_{cl})i\alpha\right]. \tag{46}$$

It changes the stationary condition Eq. (37) to

$$\frac{1}{Ng_0^2}(1-n^i_{cl}n^i_{cl})=\int\frac{d^Dp}{(2\pi)^D}\frac{e^{-\vec{p}^2/\Lambda^2}}{\vec{p}^2}=\frac{2}{D-2}\frac{\Lambda^{D-2}}{(4\pi)^{D/2}}. \tag{47}$$

Clearly this equation can be used only for $D>2$ when the integral converges with no mass term. The magnetization is then given by

$$n^i_{cl}n^i_{cl}=1-\frac{\Lambda^{D-2}}{(4\pi)^{D/2}}\frac{2Ng_0^2}{D-2}=1-\frac{g_0^2}{g_c}^2. \tag{48}$$

Therefore, one of the components needs to have an expectation value, and its size is

$$n^i_{cl}=\left(1-\frac{g_0^2}{g_c^2}\right)^{1/2}=\left(\frac{T_c-T_0}{T_c}\right)^{1/2} \tag{49}$$

as the temperature approaches the critical temperature from below. Here, we used the proportionality of the bare coupling to the temperature Eq. (25). This critical exponent $\beta = 1/2$ is the same as in the Landau theory.

At the critical point, again for $D > 2$, $\alpha = 0$ and the fields n^i are free bosons. Therefore, $\langle n^i(0) n^j(r) \rangle \propto \delta^{ij} r^{-D+2}$, and hence the critical exponent $\eta = 0$, again the same as in the Landau theory.

For the critical exponent ν, we need to distinguish three different cases, $D > 4$, $2 < D < 4$, and $D \leq 2$. The answer is *not* the same for $D < 4$ as in the Landau theory. We will discuss each case separately below.

3.1.1 $D > 4$

Obviously this is not a physically relevant case because our spatial dimensions is only three; there are systems with fewer dimensions but not more than three. Nonetheless this is a useful discussion as we can see the correspondence to the classical Landau theory of phase transitions.

Very close to the critical point, we expect the correlation length to diverge and hence $m \to 0$. We retain only the leading dependence in m^2 in the expansion Eq. (45) and the gap equation Eq. (37) becomes

$$
\begin{aligned}
& e^{m^2/\Lambda^2} \frac{\Lambda^{D-2}}{(4\pi)^{D/2}} \left(\frac{1}{\frac{D}{2}-1} - \frac{m^2/\Lambda^2}{\frac{D}{2}-2} + O(m^4) \right) \\
&= \frac{\Lambda^{D-2}}{(4\pi)^{D/2}} \left(\frac{1}{\frac{D}{2}-1} - \frac{m^2/\Lambda^2}{(\frac{D}{2}-1)(\frac{D}{2}-2)} + O(m^4) \right) = \frac{1}{N g_0^2}. \qquad (50)
\end{aligned}
$$

Using the critical coupling defined in Eq. (41), we find

$$
\frac{\Lambda^{D-2}}{(4\pi)^{D/2}} \frac{m^2/\Lambda^2}{(\frac{D}{2}-1)(\frac{D}{2}-2)} = \frac{1}{N g_c^2} - \frac{1}{N g_0^2} \qquad (51)
$$

to the leading order in m^2. Therefore, $m^2 > 0$ only for $g_0 > g_c$, and

$$
m = \left\{ \frac{(4\pi)^{D/2}}{\Lambda^{D-4}} \left(\frac{D}{2} - 1 \right) \left(\frac{D}{2} - 2 \right) \left[\frac{1}{N g_c^2} - \frac{1}{N g_0^2} \right] \right\}^{1/2}. \qquad (52)
$$

If we go back to the definition of the bare coupling Eq. (25), set $\Lambda \sim a^{-1}$, and ignore numerical factors, we find a more concrete dependence on fundamental parameters

$$\xi^{-1} = m \sim \frac{1}{a}\left(\frac{J}{T} - \frac{J}{T_c}\right)^{1/2} \sim \frac{1}{a}\left(\frac{J}{T_c}\right)^{1/2}\left(\frac{T-T_c}{T_c}\right)^{1/2}. \qquad (53)$$

Here, we set the Boltzmann constant $k = 1$, and used the fact that we are very close to the critical temperature $T \sim T_c$. Note that we introduced the correlation length

$$\langle n^i(0) n^j(\vec{x})\rangle \propto e^{-mr} = e^{-r/\xi}, \qquad (54)$$

which shows over what distance the spins are correlated.

This result makes a very good sense. For most temperatures $T > T_c$, the correlation length ξ is of the order of atomic distance a, and there is no long-range order because the two-point correlation function vanishes at large distances. However, as one approaches the critical temperature $T \to T_c$, the correlation function diverges as $(T - T_c)^{-1/2}$. At this point all spins become "fashion-conscious" and they are correlated over macroscopic distance, leading to a possible alignement of large number of spins.

This behavior $\xi \propto (T - T_c)^{-1/2}$ is precisely what one expects from Landau's theory of phase transition, as is explained in Chapter 8 of Peskin–Schroeder, in particular Eq. (8.16). This result is sometimes called mean-field theory or classical phase transition, as one can directly obtain it from the classical expression of the macroscopic free energy Eq. (8.8).

Note the connection to the running coupling constant in the expression for the correlation length. In our case, $n^i n^i = 1$ and hence there is no wave function renormalization possible, $\gamma = 0$. Therefore the Callan–Symanzik equation says

$$\Lambda \frac{D}{D\Lambda}\langle n^i(x) n^j(r)\rangle = \left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(g_0)\frac{\partial}{\partial g_0}\right]\langle n^i(x) n^j(r)\rangle = 0. \qquad (55)$$

The total derivative $D/D\Lambda$ is my notation of changing not only the cutoff but also the parameters of the theory as to not change physics. Recall Λ is the cutoff and g_0 the bare coupling defined with this cutoff. Because the two-point function depends only on m at large distances $\langle n^i(x) n^j(r)\rangle \propto e^{-mr}$, it simply is

$$\Lambda\frac{Dm}{D\Lambda} = \left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(g_0)\frac{\partial}{\partial g_0}\right] m = 0. \qquad (56)$$

On the other hand, we have worked out m in Eq. (52). Let us introduce the dimensionless coupling $\rho_0 = g_0 \Lambda^{(D-2)/2}$ in the spirit of discussion in section

12.5 of Peskin–Schroeder. They call $T = \rho_0^2$, but it is confusing with the temperature and I will not use it. Then Eq. (52) is

$$m^2 \propto \Lambda^2 \left[\frac{1}{N\rho_c^2} - \frac{1}{N\rho_0^2} \right], \tag{57}$$

where $N\rho_c^2 = (4\pi)^{D/2}(\frac{D}{2} - 1)$. Then the Callan–Symanzik equation says

$$\beta(\rho_0) = \rho_0 - \frac{\rho_0^3}{\rho_c^2} . \tag{58}$$

The coupling ρ_0 has an ultraviolet fixed point, while it can flow either to zero if $\rho_0(\Lambda) < \rho_c$ or to infinity if $\rho_0(\Lambda) > \rho_c$. The latter is the case $(T > T_c)$ where n^i acquires a finite correlation length. The finite correlation length is a non-perturbative effect because it comes with $1/g_0^2$ and hence it is possible when the coupling flows to infinity. On the other hand, for $(T < T_c)$, the coupling flows to zero and hence the low-energy limit is a weakly-coupled theory of massless n^i with the vacuum expectation value.

The Landau theory is justified for $D > 4$ in the following way. Not only the stationary condition Eq. (47), we can integrate it over $i\alpha$ and work out the effective potential Eq. (46) itself. We have

$$V_{\mathit{eff}}(\alpha, n^i_{cl}) = \frac{N}{2} \left[\frac{i\alpha}{Ng_c^2} - \frac{i\alpha}{Ng_0^2}(1 - n^i_{cl}n^i_{cl}) - \frac{\Lambda^{D/2}}{(4\pi)^{D/2}} \frac{(i\alpha)^2/\Lambda^2}{2(\frac{D}{2} - 1)(\frac{D}{2} - 2)} \right]. \tag{59}$$

Because the path integral over α is done *exactly* by substituting the stationary point thanks to the large N limit, the effective potential is then obtained as

$$\begin{aligned} V_{\mathit{eff}}(n^i_{cl}) &= N \frac{2^{D-4}(D-2)(D-4)\pi^{D/2}}{\Lambda^{D-4}} \frac{[Ng_0^2 - Ng_c^2(1 - n^i_{cl}n^i_{cl})]^2}{(Ng_0^2)^2(Ng_c^2)^2} \\ &\propto [Ng_0^2 - Ng_c^2(1 - \vec{n}^2_{cl})]^2. \end{aligned} \tag{60}$$

Recalling $g_0^2 \propto T$, $g_c^2 \propto T_c$, this is exactly the Landau theory where the potential is a quartic function of the magnetization n^i_{cl}, and the quadratic term is proportional to $T - T_c$. No wonder all the critical exponents came out to be the same as in the classical Landau theory of phase transition.

3.1.2 $2 < D < 4$

For this range of spatial dimensions, the leading term in the expansion in Eq. (45) is the first term with the Gamma function, and the gap equation

Eq. (37) is

$$\frac{\Lambda^{D-2}}{(4\pi)^{D/2}}\left[\frac{1}{\frac{D}{2}-1}+\left(\frac{m^2}{\Lambda^2}\right)^{\frac{D}{2}-1}\Gamma\left(1-\frac{D}{2}\right)\right]=\frac{1}{Ng_0^2}. \tag{61}$$

Note that for this range of D, $\Gamma(1-D/2)<0$. Again using the critical coupling Eq. (41), we find

$$\frac{\Lambda^{D-2}}{(4\pi)^{D/2}}\left|\Gamma\left(1-\frac{D}{2}\right)\right|\left(\frac{m^2}{\Lambda^2}\right)^{\frac{D}{2}-1}=\frac{1}{Ng_c^2}-\frac{1}{Ng_0^2}. \tag{62}$$

The correlation length is therefore

$$\begin{aligned}\xi^{-1}=m&=\Lambda\left\{\frac{(4\pi)^{D/2}}{\Lambda^{D-2}|\Gamma(1-D/2)|}\left[\frac{1}{Ng_c^2}-\frac{1}{Ng_0^2}\right]\right\}^{1/(D-2)}\\&\sim\frac{1}{a}\left(\frac{J}{T_c}\right)^{1/(D-2)}\left(\frac{T-T_c}{T_c}\right)^{1/(D-2)}.\end{aligned} \tag{63}$$

This is a very interesting result. The behavior of the correlation length close to the critical temperature is not what is expected from the Landau theory, but has a different power dependence on the temperature. The power $\xi\sim(T-T_c)^{-\nu}$ is called the critical exponent, and we found

$$\nu=\frac{1}{D-2}. \tag{64}$$

It agrees with $\nu=1/2$ at $D=4$ and hence is a continuous function of the dimensions. Exactly at $D=4$ the equation is a little trickier to solve because of the logarithm in Eq. (43).

Again we study the Callan–Symanzik equation. On the other hand, we have worked out m in Eq. (52). With the dimensionless coupling $\rho_0=g_0\Lambda^{(D-2)/2}$, Eq. (62) is

$$m^{D-2}\propto\Lambda^{D-2}\left[\frac{1}{N\rho_c^2}-\frac{1}{N\rho_0^2}\right], \tag{65}$$

where $N\rho_c^2=(4\pi)^{D/2}(\frac{D}{2}-1)$. We find

$$\beta(\rho_0)=\frac{D-2}{2}\left[\rho_0-\frac{\rho_0^3}{\rho_c^2}\right]. \tag{66}$$

The coupling ρ_0 has an ultraviolet fixed point, while it can flow either to zero if $\rho_0(\Lambda) < \rho_c$ or to infinity if $\rho_0(\Lambda) > \rho_c$. The latter is the case ($T > T_c$) where n^i acquires a finite correlation length. The finite correlation length is a non-perturbative effect because it comes with $1/g_0^2$ and hence it is possible when the coupling flows to infinity. On the other hand, for ($T < T_c$), the coupling flows to zero and hence the low-energy limit is a weakly-coupled theory of massless n^i with the vacuum expectation value.

The effective potential for the magnetization can be worked out the same way as the $D > 4$ case. The potential including both $i\alpha$ and n^i_{cl} is

$$V_{\mathit{eff}}(i\alpha, \vec{n}^i_{cl}) = \frac{N}{2}\left[\frac{i\alpha}{Ng_c^2} - \frac{i\alpha}{Ng_0^2}(1-\vec{n}^2_{cl}) - \left(\frac{D}{2}-1\right)\Gamma\left(-\frac{D}{2}\right)\frac{1}{Ng_c^2}\frac{(i\alpha)^{D/2}}{\Lambda^{D-2}}\right]. \tag{67}$$

Solving the stationary condition for $i\alpha$, the final effective potential is

$$\begin{aligned} V_{\mathit{eff}}(\vec{n}^i_{cl}) &= \frac{N}{2}\frac{1}{2Ng_c^2}\frac{D2^{\frac{D+2}{D-2}} - 4^{\frac{D}{D-2}}\Gamma(-\frac{D}{2})^{-\frac{D}{D-2}}}{(D(D-2))^{\frac{D}{D-2}}}\Lambda^D\left[\frac{g_0^2 - g_c^2(1-\vec{n}^2_{cl})}{g_0^2}\right]^{\frac{D}{D-2}} \\ &\propto \left[T - T_c(1-\vec{n}^2_{cl})\right]^{\frac{D}{D-2}}. \end{aligned} \tag{68}$$

This is *different* from the Landau theory. The correlation length can be easily read off from the coefficient of the quadratic term above the critical temperature $\xi^{-2} \propto (T-T_c)^{\frac{D}{D-2}-1}$ and hence $\nu = \frac{1}{D-2}$. The magnetization itself turns on below the critical temperature as $|\vec{n}_{cl}| \propto (T_c - T)^{1/2}$, which happens to be the same as in the Landau theory. On the other hand, the magnetic field at the critical temperature is

$$H = \frac{\partial V}{\partial|\vec{n}_{cl}|} \propto \frac{\partial}{\partial|\vec{n}_{cl}|}|\vec{n}_{cl}|^{\frac{2D}{D-2}} \propto |\vec{n}_{cl}|^{\frac{D+2}{D-2}}, \tag{69}$$

and hence the critical exponent $\delta = \frac{D+2}{D-2}$. The magnetization does not go linearly proportional to the applied magnetic field, but rather as $|\vec{n}_{cl}| \propto H^{1/5}$ in $D = 3$.

3.1.3 $D = 2$

The case $D = 2$ (a spin system on a planar lattice) is very special because the couping g_0 becomes dimensionless. Solving Eq. (44) is easy,

$$\xi^{-1} = m = \Lambda e^{-2\pi/Ng_0^2}e^{-\gamma/2}. \tag{70}$$

Note that this is a solution for *any* values of the bare coupling, and hence the correlation length is finite for *all* temperatures. What it means is that there is *never* a true long-range order of spontaneously broken $SO(N)$ symmetry in two dimensions. This is in accord with the Mermin–Wagner Theorem that states that a continuous symmetry cannot be spontaneously broken in two or fewer dimensions for any finite temperature. However the correlation length can be much much longer than the atomic distance because of the exponentially factor, which goes as $e^{c/T}$ for a constant c. Even though the behavior $T \to 0$ cannot be studied rigorously with our method because we integrated out the non-zero modes in the imaginary time direction, we see that the zero-temperature limit can break the symmetry. The ground state is still that of aligned spins. This does not contradict Mermin–Wagner theorem because the field theory is three-dimensional at $T = 0$.

Now we study the connection to the running coupling constant in the expression for the correlation length. Using m in Eq. (70). We find

$$\beta(g_0) = \Lambda \frac{\partial g_0}{\partial \Lambda} = -\frac{N g_0^3}{4\pi} < 0. \tag{71}$$

The negative beta function means the theory is asymptotically free. It is much easier to deal with a different form,

$$\Lambda \frac{\partial}{\partial \Lambda} \frac{1}{g_0^2} = \frac{N}{2\pi}. \tag{72}$$

Namely, when we integrate out the momentum slice to bring the cutoff Λ down to $\Lambda' = b\Lambda$ $(0 < b < 1)$, we make the bare coupling bigger accordingly,

$$\frac{1}{g_0^2(\Lambda')} = \frac{1}{g_0^2(\Lambda)} - \ln \frac{\Lambda}{\Lambda'}, \tag{73}$$

or

$$g_0^2(\Lambda') = \frac{1}{g_0^{-2}(\Lambda) - \frac{N}{2\pi} \ln(\Lambda/\Lambda')}. \tag{74}$$

When it is integrated down to $\Lambda' = \Lambda e^{-2\pi/N g_0^2(\Lambda)}$, the coupling becomes infinitely strong and the perturbation theory breaks down. Thanks to the exact solution, we nonetheless know what happens. Despite the theory having no dimensionless parameter, it develops a definite mass scale where the theory becomes strong and produces a finite correlation length. This is the phenomenon called dimensional transmutation. In Wilsonian way of looking at it, is it not too much of a surprise; there is a definite ultraviolet cutoff to the theory where the theory is defined. The surprise is rather on the emergence of an exponentially smaller mass scale (larger distance scale) compared to the cutoff, without finely tuning the bare parameters which was required in theories $D > 2$.

The spins are more-or-less lined up together over a some mesoscopic distance, but macroscopically they are random. The critical temperature is formally at $T_c = 0$ where the ground state still breaks the symmetry. For any finite temperatures, however, the symmetry is not broken.

The effective potential can be computed as in the previous cases, and we obtain

$$V_{\mathit{eff}}(\vec{n}_{cl}) = \frac{N}{8\pi}\Lambda^2 e^{-\gamma-4\pi/Ng_0^2} e^{4\pi\vec{n}_{cl}^2/Ng_0^2}. \tag{75}$$

Together with the kinetic term $\frac{1}{2g_0^2}(\vec{\nabla} n^i_{cl})^2$, it is easy to read off the correlation length $\xi^{-2} = m^2 = \Lambda^2 e^{-\gamma-4\pi/Ng_0^2}$. The response to the finite magnetic field is highly non-trivial with the exponential dependence.

3.1.4 $D = 1$

For $D = 1$, the gap equation Eq. (37) becomes

$$\frac{1}{2m} - \frac{1}{\sqrt{\pi}\,\Lambda} = \frac{1}{Ng_0^2} . \tag{76}$$

Therefore,

$$\xi = m^{-1} = \frac{2}{Ng_0^2} + \frac{2}{\sqrt{\pi}\,\Lambda} . \tag{77}$$

Again there is a solution for any values of g_0 and hence there is no long-range order. In addition, the correlation length is very short. Using the bare coupling in Eq. (25), we find

$$\xi = j^2 a \frac{2}{N}\frac{J}{T} + \frac{2}{\sqrt{\pi}\,\Lambda} , \tag{78}$$

and hence of the order of the atomic distance in general. Again as $T \to 0$, the correlation length does become large, but in this case it goes like power and a macroscopic correlation would require the temperature exponentially close to zero. The ground state theoretically breaks symmetry, yet practically never does.

3.2 Perturbation Theory around $D = 2$

In the previous section, we solved the Heisenberg model in all dimensions in the large N limit. We did not rely on perturbation theory and hence the result was powerful and dramatic. On the other hand, we cannot study the realistic system with $N = 3$.

At $D = 2$ the non-linear sigmal model is renormalizable. Even though there is nothing wrong with non-renormalizable field theories, the results are sensitive to the physics at the cutoff scale. On the other hand, with renormalizable field theories you can work out all physical quantities using the "measurements" close to the energy scale you have probed and pretend that the cutoff scale is infinite. This provides an extra comfort in the results you work out with perturbation theory.

To use the perturbation theory, we need to solve the constraint $\sum_{i=1}^N n^i n^i = 1$ explicitly as

$$n^i = (\pi^1, \cdots, \pi^{N-1}, (1 - \pi^i \pi^i)^{1/2}). \tag{79}$$

The action is then

$$S = \frac{1}{2g_0^2} \int d^D x \left[\sum_{i=1}^{N-1} (\vec{\nabla} \pi^i)^2 + \frac{(\pi^i \vec{\nabla} \pi^i)^2}{1 - \pi^i \pi^i} \right]. \tag{80}$$

Changing the normalizaton $\pi^i \to g_0 \pi^i$, we find

$$S = \frac{1}{2} \int d^D x \left[\sum_{i=1}^{N-1} (\vec{\nabla} \pi^i)^2 + g_0^2 \frac{(\pi^i \vec{\nabla} \pi^i)^2}{1 - g_0^2 \pi^i \pi^i} \right]. \tag{81}$$

In power series in g_0, we can develop perturbation theory.

The actual computation is described in detail in the book, and let me just highlight the important results in $D = 2$. First of all, the beta function is

$$\beta(g) = -(N-2) \frac{g^3}{4\pi} + O(g^3). \tag{82}$$

This is consistent with the exact result for large N, $\beta(g) = -N \frac{g^3}{4\pi}$. In particular, it is asymptotically free and it generates a finite correlation length no matter how small g_0 is due to the dimensional transmutation. (Strictly speaking, this conclusion cannot be proven from perturbation theory as the higher order terms could in princple lead the coupling to an infrared fixed point. However the Mermin–Wagner theorem suggests this must be the case, and the perturbation theory supports it. For large N, we could *reallly* show the finite correlation length even though it was beyond the validity of the perturbation theory.)

Second, the beta function vanishes for $N = 2$. This makes sense because for $N = 2$ we can parameterize $n^i = (\cos\theta, \sin\theta)$, and the action is that of a free scalar,

$$S = \frac{1}{2g_0^2} \int d^2x (\vec{\nabla}\theta)^2. \tag{83}$$

Therefore, the beta function vanishes to all orders in perturbation theory. However, even this free theory shouldn't be underestimated. The spin-spin correlation function for $s(\vec{x}) = \pi^1 + i\pi^2$ can be worked out (see Problem 11.1) that shows a power law behavior

$$\langle s(0)s^*(\vec{x})\rangle \propto |\vec{x}|^{-cg_0^2}, \tag{84}$$

where c is a numerical constant. Recall $g_0^2 \propto T$ and for any finite temperature the correlation vanishes at very large distances. This is another manifestation of the Mermin–Wagner theorem that continuous symmetries (in this case it is a rotation among two components $SO(2)$) cannot be broken in two dimensions at a finite temperature.

For $D = 2 + \epsilon$ but $\epsilon \ll 1$, we can still apply the perturbation theory in the following way. Peskin–Schroeder define $T = g_0^2\Lambda^{D-2}$ as a dimensionless coupling. Indeed, this combination is proportional to the temperature even though it may not be exactly be the same. For this dimensionless coupling, the beta function is

$$\beta(T) = (D-2)T - (N-2)\frac{T^2}{2\pi} + O(T^3). \tag{85}$$

There is an ultraviolet fixed point

$$T_* = \frac{2\pi\epsilon}{N-2}, \tag{86}$$

which is small enough to trust the perturbation theory is ϵ is small. Then using the perturbative calculation of anomalous dimension factors, one arrives at the predictions

$$\eta = \frac{\epsilon}{N-2}, \qquad \nu = \frac{1}{\epsilon}. \tag{87}$$

They are consistent with the large N result $\eta = 0$ and $\nu = \frac{1}{D-2}$.

4 $O(N)$ Linear Sigma Model

To understand the behavior of the theory near $D = 4$, a different description of the theory is suitable, based on linear sigmal models. This is because it is renormalizable and perturbation theory is worked out rigorously without referring to the cutoff-scale quantities.

This model is called linear sigma model because of the following reason. Consider the ϕ^4 theory with the symmetry-breaking potential

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^i \partial^\mu \phi^i - \frac{\lambda}{4}(\phi^i\phi^i - v^2)^2. \tag{88}$$

At the minimum of the potential, $\phi^i\phi^i = v^2$. Choosing

$$\langle \phi^i \rangle = (0, \cdots, 0, v), \tag{89}$$

$SO(N)$ symmetry is broken to $SO(N-1)$. To understand the excitation spectrum of the theory, the book parameterizes the field as

$$\phi^i = (\pi^1, \cdots, \pi^{N-1}, v + \sigma). \tag{90}$$

The π^i fields are massless Nambu–Goldstone bosons that describe the fluctuation of the vacuum along the bottom of the potential, while σ is a massive field that describes the fluctuation of vacuum around the bottom of the potential. Because of the notation σ for this mode, it is called sigma model, while the kinetic term for π^i and σ are the usual ones $\frac{1}{2}(\partial\pi^i)^2 + \frac{1}{2}(\partial\sigma)^2$. The fields span the linear space $\mathbb{R}^N$ and hence the *linear* sigma model.

On the other hand, we can also parametrize the N scalar fields as

$$\phi^i = (v + \sigma)n^i, \qquad n^i n^i = 1. \tag{91}$$

Noting $n^i\partial_\mu n^i = \frac{1}{2}\partial_\mu(n^i n^i) = 0$, we find

$$\begin{aligned} \partial_\mu\phi^i\partial^\mu\phi^i &= (n^i\partial_\mu\sigma + (v+\sigma)\partial_\mu n^i)(n^i\partial^\mu\sigma + (v+\sigma)\partial^\mu n^i) \\ &= \partial_\mu\sigma\partial^\mu\sigma + (v+\sigma)^2\partial_\mu n^i\partial^\mu n^i. \end{aligned} \tag{92}$$

Hence the Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}(v+\sigma)^2\partial_\mu n^i\partial^\mu n^i - \frac{\lambda}{4}(2v\sigma + \sigma^2)^2. \tag{93}$$

Therefore, σ is a field of finite mass $2\lambda v^2$, while n^i are $N-1$ massless Nambu–Goldstone bosons (remember the constraint $n^i n^i = 1$ removes one of the components). The non-linear sigma model is obtained by taking the limit $\lambda \to \infty$, where the sigma becomes infinitely massive and can be integrated out from the theory. The field content n^i not longer spans a linear space but rather a non-linear space S^{N-1}, hence *non-linear* sigma model. It is rather strange that it is called non-linear **sigma** model when σ is actually removed from the theory, but it is just a historical name.

In general, when the symmetry G breaks to $H \subset G$, the remaining degrees of freedom (Nambu–Goldstone bosons) span the coset space G/H. In the case above, it is $SO(N)/SO(N-1)$, which is nothing but S^{N-1}. The term non-linear sigma model is used even more broadly, referring to scalar field theory where the fields span any non-linear space with the metric $ds^2 = g_{ij}(x)dx^i dx^j$. The Lagrangian density is then given by $\mathcal{L} = \frac{1}{2}g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j$.

The linear sigma model is relevant for us because it can be regarded as the description of low-enegy behavior of the non-linear sigma model. Recall that in $D = 4$ the beta function for the linear sigma model is positive, and hence the coupling goes to infinite at some high energy scale (Landau pole). We can identify the Landau pole as the cutoff where the non-linear sigma model is defined (atomic scale). As one integrate out momentum slices, the coupling flows to smaller and smaller values and at some point the perturation theory becomes good. This way, the linear sigma model emerges from the non-linear sigma model. In other words, the potential

$$V = \frac{\lambda}{4}(\phi^i\phi^i - v^2)^2 \qquad (94)$$

is infinitely steep to enforce the constraint $\phi^i\phi^i = v^2$ at the cutoff, while it is more *relaxed* at lower energies because of the "block spin" procedure that averages out the spins. Therefore, we can use the linear sigma model for our study.

Again the book describes in detail how to work out various critical exponents using the solutions to the Callan–Symanzik equation. Here I summarize only the highlights.

First of all, the beta function for the coupling λ in $D = 4 - \epsilon$ dimensions has an *infrared* fixed point,

$$\beta(\lambda) = (N+8)\frac{\lambda^2}{8\pi^2} - \epsilon\lambda. \qquad (95)$$

Now matter what values of λ the theory starts out at the cutoff, it flows to the fixed point as long as we study the behavior at large enough distances. The fixed-point coupling is

$$\lambda_* = \frac{8\pi^2}{N+8}\epsilon. \tag{96}$$

The perturbation theory is trustworthy when λ_* is small, and hence small ϵ. The fixed-point is extremely useful as the theory becomes predictive. Independent of the details of the cutoff scale theory, we can work out predictions at the fixed point. The only remaining free parameter is the coefficient of the mass operator ϕ^2, which is proportional to $T - T_c$. The cutoff-scale theory even doesn't have to be Heisenberg model anymore as long as it leads to the linear sigma model at long distances; they lead to the same predictions. This remarkable simplification of long-distance behavior close to the critical point is called *universality*, the basis of all studies of critical phenomena.

Here is a table that shows the consistency between the large N exact results for non-linear sigma model and finite N result for linear sigma model close to four dimensions.

critical exponents	large N	$D = 4 - \epsilon$
ν^{-1}	$D - 2$	$2 - \gamma_{\phi^2}(\lambda_*) = 2 - \frac{N+2}{N+8}\epsilon$
η	0	$2\gamma(\lambda_*) = \frac{N+2}{2(N+8)^2}\epsilon^2$
β	$\frac{1}{2}$	$\frac{1}{2}(D - 2 + \eta)\nu = \frac{1}{2}(1 - \frac{3}{N+8}\epsilon)$
α	$\frac{D-4}{D-2}$	$2 - D\nu$
δ	$\frac{D+2}{D-2}$	$\frac{D+2-2\gamma(\lambda_*)}{D-2+2\gamma(\lambda_*)}$

Here,

$$\gamma(\lambda_*) = (N+2)\frac{\lambda_*^2}{4(8\pi^2)^2} = \frac{N+2}{4(N+8)^2}\epsilon^2, \quad \gamma_{\phi^2}(\lambda_*) = (N+2)\frac{\lambda_*}{8\pi^2} = \frac{N+2}{N+8}\epsilon \tag{97}$$

to this order in perturbation theory.

5 $D = 3$, Finite N

Working out quantitative predictions for $D = 3$ and finite N is a big challenge. The IR fixed point still persists, but the fixed-point coupling is large

$\lambda_* \sim 8\pi^2$ and hence the fixed-order perturbation theory may not be trusted. Therefore, we need to come up with a clever way of summing the perturbation series and then extrapolate it to $\epsilon = 1$.

Another complication is that perturbation series has *zero* radius of convergence. Yes, zero. There is a very simple way to understand why, as explained in a separate lecture notes written for 221A. The main part of the argument is this. If the perturbation series in λ has a finite radius of convergence, it should converge also for a small negative value. However, for a negative λ, the potential is not bounded from below, and there is a tunneling from $\phi = 0$ to $\phi = \infty$. The tunneling amplitude estimated by the WKB method goes as $e^{-c/|\lambda|}$ for a constant c. Taylor expansion of this amplitude to any finite orders in $|\lambda|$ around $\lambda = 0$ vanishes because of the exponential factor. Therefore, the perturbation series cannot have a finite radius of convergence, and hence should be regarded as an asymptotic series.

There is a general prescription called Borel summation to deal with an asymptotic series, sum the first finite number of terms, and estimate the entire series. Suppose you are computing a perturbation series

$$A(g) = \sum_{k=0}^{\infty} A_k g^k. \tag{98}$$

The problem is that the high-order terms misbehave, $A_k \sim c k^{b_0} (-a)^k k! (1 + O(k^{-1}))$ as $k \to \infty$. This is the behavior one can identify even in a simple integral (*i.e.*, zero-dimensional quantum field theory)

$$\int_{-\infty}^{\infty} dx\, e^{-x^2 - g x^4}, \tag{99}$$

if you expand it in power series in g. Instead of trying to sum this series that diverges anyway, one defines

$$B_b(g) = \sum_{k=0}^{\infty} \frac{A_k}{\Gamma(k+b+1)} g^k. \tag{100}$$

Because of the Gamma function in the denominator, this series converges. Then you perform a Laplace transform,

$$A(g) = \int_0^{\infty} dt\, t^b e^{-t} B_b(gt). \tag{101}$$

If the series converges uniformly, one can interchange the sum and the integral, and one can see the equivalence very easily. However for an asymptotic series, this method (Borel sum) converges while the simple perturbative series does not.

In the paper J. C. Le Guillou and J. Zinn-Justin, "Critical exponents from field theory," *Phys. Rev.* **B 21**, 3976-3998 (1980), they employ the Borel sum to study the linear sigma model in $D = 3$. The results are reproduced in Table 13.1 in Peskin–Schroeder (with some updates) on page 450. The agreement with the experiments is remarkable.

6 Conformal Field Theories

The field theories at the fixed points have long-range correlations yet with funny powers (critical exponents). These theories are called "conformal field theories." First of all, the theories at the critical points are "scale-invariant" because all correlation functions follow power laws and hence there is no dimensionful parameter such as mass. Classically non-linear sigma models in $D = 2$ are scale invariant, so are non-abelian gauge theories with no quarks or massless quarks. However, these theories develop scale dependence due to the renormalization condition (or UV cutoff) and are not scale-invariant quantum mechanically. On the other hand, theories with apparent mass scales can flow to scale-invariant theories in the infrared. It turns out that most scale-invariant theories are also *conformally* invariant. Conformal group is generated by the usual Lorentz generators

$$M_{\mu\nu} = -i(x_\mu \partial_\nu - x_\nu \partial_\mu), \tag{102}$$

translation generators

$$P_\mu = -i\partial_\mu \tag{103}$$

together with two new sets of generators for dilation (overall scale change)

$$D = -x_\mu \partial^\mu \tag{104}$$

and the special conformal transformations

$$K_\mu = -\frac{i}{2}(x^2 \partial_\mu - 2x_\mu x_\nu \partial^\nu). \tag{105}$$

K_μ generate transformations which are a combination of inversion $x^\mu \to x^\mu/x^2$, then translation $x^\mu/x^2 + k^\mu$, and the inversion back

$$\begin{aligned}\frac{x^\mu/x^2 + k^\mu}{(x^\nu/x^2 + k^\nu)^2} &= \frac{x^\mu/x^2 + k^\mu}{1/x^2 + 2(k_\nu x^\nu)/x^2 + k^2} \\ &= \frac{x^\mu + k^\mu x^2}{1 + 2(k \cdot x) + k^2 x^2} \\ &= x^\mu - 2(k \cdot x)x^\mu + k^\mu x^2 + O(k^2) \\ &= x^\mu + k_\nu(2x^\nu x^\mu \partial_\nu - x^2 \partial_\nu)x^\mu + O(k^2) \\ &= (1 - 2ik_\nu K^\nu)x^\mu + O(k^2). \end{aligned} \tag{106}$$

Together, a conformal group in D-dimensional Minkowski space forms the $SO(D,2)$ symmetry. Similarly a conformal group in D-dimensional Euclidean space forms the $SO(D+1,1)$ symmetry.

In two dimensions, the conformal group can be extended far beyond $SO(3,1)$ symmetry to an infinite-dimensional symmetry based on the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12\pi} n(n^2-1)\delta_{n,-m}. \tag{107}$$

Because of this powerful symmetry, one can classify the possible conformal field theories and predict their properties in great detail.

For example, the critical point of the Ising model is described by one of the so-called minimal models of two-dimensional conformal field theory. This identification allows one to compute the correlation functions *exactly* at the critical point without relying on perturbation theory. Critical behavior of many other statistical systems have been studied with conformal field theory.