Notes on Clifford Algebra and Spin(N) Representations

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1 Euclidean Space

We first consider representations of Spin(N).

1.1 Clifford Algebra

The Clifford Algebra is

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij}. \tag{1}$$

The point of studying Clifford algebra is that once you find representations of Clifford algebra you can immediately construct representations of Spin(N). Generators of rotations in Spin(N) are given by

$$M^{ij} = \frac{i}{4}[\gamma^i, \gamma^j]. \tag{2}$$

It is easy to see that they satisfy the Lie algebra of SO(N).

A representation of Clifford algebra can be constructed by tensor products of $[N/2]$ Pauli matrices. For even dimensions $N = 2k$, we have k Pauli matrices,

$$\gamma^1 = \sigma_1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1, \tag{3}$$
$$\gamma^2 = \sigma_2 \otimes 1 \otimes \cdots \otimes 1 \otimes 1, \tag{4}$$
$$\gamma^3 = \sigma_3 \otimes \sigma_1 \otimes \cdots \otimes 1 \otimes 1, \tag{5}$$
$$\gamma^4 = \sigma_3 \otimes \sigma_2 \otimes \cdots \otimes 1 \otimes 1, \tag{6}$$
$$\vdots$$
$$\gamma^{2k-3} = \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_1 \otimes 1, \tag{7}$$
$$\gamma^{2k-2} = \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_2 \otimes 1, \tag{8}$$
$$\gamma^{2k-1} = \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_1, \tag{9}$$
$$\gamma^{2k} = \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_2, \tag{10}$$
$$\gamma^{2k+1} = \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_3. \tag{11}$$

The last one γ^{2k+1} plays the role of γ^5 in four dimensions. You can readily verify

$$\gamma^{2k+1} = (-i)^k \gamma^1 \gamma^2 \cdots \gamma^{2k-1} \gamma^{2k}. \tag{12}$$

It is easy to see that any of two gamma matrices anti-commute, while the square of any one is an identity matrix. Therefore, this gives a representation of Clifford algebra for Spin($2k$). In fact, This is a representation of Clifford algebra for Spin($2k+1$) as well, by including γ^{2k+1}.

1.2 Reality Property of General Representations

Here are a few remarks about the reality property of a representation of compact simple Lie algebra.

When given an irreducible representation matrix $\rho(T^a)$ for generators T^a, the "conjugate representation" is defined by $-\rho(T^a)^T$. It is easy to check that they satisfy the same Lie algebra.* The reason this is called a conjugate representation is because the representation matrix of a group element is simply the complex conjugate,

$$\rho(g) = \rho(e^{i\omega^a T^a}) = e^{i\omega^a \rho(T^a)} \to e^{-i\omega^a \rho(T^a)^T} = e^{-i\omega^a \rho(T^a)*} = \rho(g)^*. \tag{13}$$

I used the hermiticity of the generators $\rho(T^a)^T = \rho(T^a)^*$.†

A representation is said to be real if you can find a unitary matrix C that satisfies

$$\rho(T^a)^T = -C\rho(T^a)C^{-1}. \tag{14}$$

Then the representation and its conjugate representation are unitary equivalent. If there is no such matrix C, the representation is said to be complex.

Taking the transpose of the both sides,

$$\rho(T^a) = -C^{-1T} \rho(T^a)^T C^T. \tag{15}$$

Substituting it back into Eq. (14),

$$\rho(T^a)^T = -C(-C^{-1T}\rho(T^a)^T C^T)C^{-1} = (CC^{-1T})\rho(T^a)^T (CC^{-1T})^{-1}. \tag{16}$$

*There are many conjugations you can define. The definition here is what is commonly used in the physics literature.

†This of course breaks down for Lorentz groups because they are non-compact and hence their finite-dimensional representations are non-unitary.

In other words, the matrix CC^{-1T} commutes will all generators. According to Schur's lemma, such a matrix must be proportional to an identity matrix for an irreducible representation. Therefore, $CC^{-1T} = c1$, or

$$C = cC^T. \tag{17}$$

Taking transpose of both sides, we find $C^T = cC = c^2 C^T$, and hence $c^2 = 1$. Only possibilities are $c = \pm 1$, and hence C is either symmeric or anti-symmetric matrix.

Under a general unitarity transformation of generators $\rho(T^a)' = U^\dagger \rho(T^a) U$, the Eq. (14) changes to

$$\rho(T^a)'^T = U^T \rho(T^a) U^* = -U^T C \rho(T^a) C^{-1} U^* = -U^T C U \rho(T^a)' U^\dagger C^{-1} U^*. \tag{18}$$

Therefore in the new basis, the matrix C becomes $U^T C U$. As a result, the symmetry of C, whether it is symmetric $C = C^T$ or anti-symmetric $C = -C^T$, is basis-independent.

There is still a question if you can find a basis where all representation matrices of group elements are real. If you cannot find such a basis, the representation is said to be pseudo-real. If you can, it is real in the strict sense.

Any symmetric complex matrix $A^T = A$ can be transformed to a real semi-positive diagonal matrix $U^T A U = D$. If C is symmetric, we can therefore find a basis where C is real semi-positive diagonal. But C is unitary, and hence eigenvalues are phases. The only real semi-positive phase is unity. Therefore in this basis the matrix C is a unit matrix, and we find

$$\rho(T^a)^T = -C\rho(T^a) C^{-1} = -\rho(T^a), \tag{19}$$

so that all generators are represented as pure imaginary anti-symmetric matrices. Then the representation of group elements is real:

$$\rho(g)^* = e^{-i\omega^a \rho(T^a)^T} = e^{i\omega^a \rho(T^a)} = \rho(g). \tag{20}$$

You can see that all representation matrices are real if C is symmetric.

On the other hand, if C is anti-symmetric, there is no basis where C is an identity matrix which is symmetric. In this case, the representation and its conjugate representation are unitarity equivalent, but you cannot find a basis where the representation matrices are real. The doublet representation of SU(2), or the fundamental representations of Sp(N) groups are examples of such pseudo-real representations.

1.3 Reality Property of Clifford Algebra

In the case of Clifford algebra, γ-matrices are roughly speaking square root of group generators, and there are two possibilities for the representation to qualify as "real",[‡]

$$\rho(T^a)^T = -\mathcal{C}\gamma^i\mathcal{C}^{-1}, \tag{21}$$
$$\rho(T^a)^T = +\mathcal{T}\gamma^i\mathcal{T}^{-1}. \tag{22}$$

If $\mathcal{C}$ exists, you can make all γ-matrices pure imaginary anti-symmetric, following the same arguments as the group generators in the previous section. On the other hand, if $\mathcal{T}$ exists, you can make all γ-matrices pure real symmetric.

Given the explicit representation of Clifford algebra in Eq. (11), we see that γ^i are symmetric for odd i, and anti-symmetric for even i. We somehow need to distinguish even and odd gammas. Therefore we try matrices

$$\mathcal{C}_1 = \gamma^1\gamma^3\cdots\gamma^{2k-1}, \tag{23}$$
$$\mathcal{C}_2 = \gamma^2\gamma^4\cdots\gamma^{2k}. \tag{24}$$

From this point on, the situation depends on k mod 4 (hence N mod 8).

1.3.1 $k = 0$ mod 4

In this case,

$$\mathcal{C}_1 = (-i\sigma_2)\otimes\sigma_1\otimes\cdots(-i\sigma_2)\otimes\sigma_1, \tag{25}$$
$$\mathcal{C}_2 = (i\sigma_1)\otimes\sigma_2\otimes\cdots(i\sigma_1)\otimes\sigma_2. \tag{26}$$

From the definitions Eqs. (23,24), we find

$$\mathcal{C}_1\gamma^i\mathcal{C}_1^{-1} = -\gamma^{iT}, \tag{27}$$
$$\mathcal{C}_2\gamma^i\mathcal{C}_2^{-1} = \gamma^{iT}. \tag{28}$$

Hence, $\mathcal{C} = \mathcal{C}_1$, $\mathcal{T} = \mathcal{C}_2$.

Both $\mathcal{C}$ and $\mathcal{T}$ are symmetric.

Both $\mathcal{C}$ and $\mathcal{T}$ commute with γ^{2k+1}.

[‡]The matrix $\mathcal{C}$ corresponds to charge conjugation, while $\mathcal{T}$ to time reversal in 3+1 dimensional Dirac equation, hence this notation.

1.3.2 $k = 1$ mod 4

In this case,

$$\begin{aligned} \mathcal{C}_1 &= \sigma_1 \otimes (-i\sigma_2) \otimes \cdots (-i\sigma_2) \otimes \sigma_1, & (29) \\ \mathcal{C}_2 &= \sigma_2 \otimes (i\sigma_1) \otimes \cdots (i\sigma_1) \otimes \sigma_2. & (30) \end{aligned}$$

From the definitions Eqs. (23,24), we find

$$\begin{aligned} \mathcal{C}_1 \gamma^i \mathcal{C}_1^{-1} &= \gamma^{iT}, & (31) \\ \mathcal{C}_2 \gamma^i \mathcal{C}_2^{-1} &= -\gamma^{iT}. & (32) \end{aligned}$$

Hence, $\mathcal{C} = \mathcal{C}_2$, $\mathcal{T} = \mathcal{C}_1$.

$\mathcal{C}$ is anti-symmetric while $\mathcal{T}$ is symmetric.

Both $\mathcal{C}$ and $\mathcal{T}$ anti-commute with γ^{2k+1}.

1.3.3 $k = 2$ mod 4

In this case,

$$\begin{aligned} \mathcal{C}_1 &= (-i\sigma_2) \otimes \sigma_1 \otimes \cdots (-i\sigma_2) \otimes \sigma_1, & (33) \\ \mathcal{C}_2 &= (i\sigma_1) \otimes \sigma_2 \otimes \cdots (i\sigma_1) \otimes \sigma_2. & (34) \end{aligned}$$

From the definitions Eqs. (23,24), we find

$$\begin{aligned} \mathcal{C}_1 \gamma^i \mathcal{C}_1^{-1} &= -\gamma^{iT}, & (35) \\ \mathcal{C}_2 \gamma^i \mathcal{C}_2^{-1} &= \gamma^{iT}. & (36) \end{aligned}$$

Hence, $\mathcal{C} = \mathcal{C}_1$, $\mathcal{T} = \mathcal{C}_2$.

Both $\mathcal{C}$ and $\mathcal{T}$ are anti-symmetric.

Both $\mathcal{C}$ and $\mathcal{T}$ commute with γ^{2k+1}.

1.3.4 $k = 3$ mod 4

In this case,

$$\begin{aligned} \mathcal{C}_1 &= \sigma_1 \otimes (-i\sigma_2) \otimes \cdots (-i\sigma_2) \otimes \sigma_1, & (37) \\ \mathcal{C}_2 &= \sigma_2 \otimes (i\sigma_1) \otimes \cdots (i\sigma_1) \otimes \sigma_2. & (38) \end{aligned}$$

From the definitions Eqs. (23,24), we find

$$\mathcal{C}_1\gamma^i\mathcal{C}_1^{-1} = \gamma^{iT}, \quad (39)$$
$$\mathcal{C}_2\gamma^i\mathcal{C}_2^{-1} = -\gamma^{iT}. \quad (40)$$

Hence, $\mathcal{C} = \mathcal{C}_2$, $\mathcal{T} = \mathcal{C}_1$.

$\mathcal{C}$ is symmetric while $\mathcal{T}$ is anti-symmetric.

Both $\mathcal{C}$ and $\mathcal{T}$ anti-commute with γ^{2k+1}.

1.3.5 Summary for Even Dimensions

Summarizing discussions, the properties of $\mathcal{C}$ and $\mathcal{T}$ matrices are given in the Table below.

N	$\mathcal{C}$		$\mathcal{T}$	
	S/A	C/A	S/A	C/A
$8k$	S	C	S	C
$8k+2$	A	A	S	A
$8k+4$	A	C	A	C
$8k+6$	S	A	A	A

Table 1: Properties of $\mathcal{C}$ and $\mathcal{T}$ matrices for various k. S/A refers to symmetric or anti-symmetric. C/A refers to either commute or anti-commute with γ^{2k+1}.

For $N = 8k$ and $N = 8k+4$ dimensions, both $\mathcal{C}$ and $\mathcal{T}$ matrices commute with γ^{N+1}. Therefore, the irreducible representations of Spin(N) with definite chirality under γ^{N+1} are each self-conjugate. For $N = 8k$ $\mathcal{C}, \mathcal{T}$ are symmetric and the irreducible representations are both real in the strict sense. They are called Majorana–Weyl spinors.

1.3.6 Summary for Odd Dimensions

For odd $(2k+1)$ dimensions, we include γ^{2k+1} as one of the gamma matrices. Not both $\mathcal{C}$ and $\mathcal{T}$ matrices exist for odd dimensions.

For $8n$ dimensions, both $\mathcal{C}$ and $\mathcal{T}$ commute with γ^{2k+1}. Therefore there is no $\mathcal{C}$ matrix for $8n+1$ dimensions. $\mathcal{T}$ matrix is symmetric.

For $8n+2$ dimensions, both $\mathcal{C}$ and $\mathcal{T}$ anti-commute with γ^{2k+1}. Therefore there is no $\mathcal{T}$ matrix for $8n+3$ dimensions. $\mathcal{C}$ matrix is anti-symmetric.

For $8n+4$ dimensions, both $\mathcal{C}$ and $\mathcal{T}$ commute with γ^{2k+1}. Therefore there is no $\mathcal{C}$ matrix for $8n+5$ dimensions. $\mathcal{T}$ matrix is anti-symmetric.

For $8n+6$ dimensions, both $\mathcal{C}$ and $\mathcal{T}$ anti-commute with γ^{2k+1}. Therefore there is no $\mathcal{T}$ matrix for $8n+7$ dimensions. $\mathcal{C}$ matrix is anti-symmetric.

N	$\mathcal{C}$	$\mathcal{T}$
$8k+1$	N/A	S
$8k+3$	A	N/A
$8k+5$	N/A	A
$8k+7$	S	N/A

Table 2: Properties of $\mathcal{C}$ and $\mathcal{T}$ matrices for various odd N. S/A refers to symmetric or anti-symmetric. N/A means it does not exist.

1.4 Majorana Spinor

Dirac equation in Euclidean space is[§]

$$(\gamma^i(\partial_i - igA_iT^a) - m)\psi = 0. \tag{41}$$

Here, A_i is a gauge field, and T^a the (hermitian) representation matrices of the corresponding Lie algebra. An interesting question is if this equation is real, so that we can consistently impose reality on ψ. If $\mathcal{T}$ exists for that dimension and is symmetric, γ^i can be taken to be all real, and hence $\gamma^i\partial_i$ is real. The whole equation then becomes real if T^a are all pure imaginary, *i.e.* if the fermion belongs to a real representation of the gauge group. Therefore, a reality condition can be imposed on ψ if all γ-matrices can be taken real and if the fermion belongs to a real representation of the gauge group. Such a fermion is called Majorana spinor. In a general representation where γ-matrices are not taken to be real, the complex conjugate of Dirac equation is

$$\begin{aligned}
&(\gamma^{i*}(\partial_i + igA_iT^{a*}) - m)\psi^* \\
&\quad = (\mathcal{T}\gamma^i\mathcal{T}^{-1}(\partial_i + igA_iT^{a*}) - m)\psi^* \\
&\quad = \mathcal{T}(\gamma^i(\partial_i - igA_iT^a) - m)\mathcal{T}^{-1}\psi^* = 0.
\end{aligned} \tag{42}$$

[§]Dirac equation does not have a solution in Euclidean space, unless $m=0$ and there exists a zero mode for γ^iD_i. Nor does Klein–Gordon equation $(\partial^2 - m^2)\phi = 0$.

In the last line, we assumed a real representation of the gauge group. The Majorana condition is then $\psi^* = \mathcal{T}\psi$. Taking complex conjugate of both sides, $\psi = \mathcal{T}^*\psi^* = \mathcal{T}^*\mathcal{T}\psi$. If $\mathcal{T}$ is anti-symmetric, $\mathcal{T}^*\mathcal{T} = -\mathcal{T}^\dagger\mathcal{T} = -1$, and Majorana condition is inconsistent. However, if $\mathcal{T}$ is symmetric, this condition is consistent and indeed can be imposed on spinors.

Note that having a Majorana spinor is a stronger condition than just having real representation of Spin(N). For $8k+7$ dimensions, there is a symmetric $\mathcal{C}$ matrix and hence all gamma-matrices can be taken pure imaginary. This makes the Spin($8k+7$) representation real. However, pure imaginary gamma-matrices do not make Dirac equation real, and we cannot impose Majorana condition on the spinor.

Only for $N = 8k$ dimensions, $\mathcal{T}$ is symmetric and commutes with γ^{N+1}. Therefore, the Majorana condition can be satisfied for a spinor with a definite chirality. Such a fermion is called Majorana–Weyl spinor.

Combining results from previous sections, here is the table that shows in what dimensions Majorana or Majorana–Weyl spinor can exist.

N	Weyl	reality	Majorana	Majorana–Weyl
$8k$	yes	real	yes	yes
$8k+1$	no	real	yes	no
$8k+2$	yes	complex	yes	no
$8k+3$	no	pseudo-real	no	no
$8k+4$	yes	pseudo-real	no	no
$8k+5$	no	pseudo-real	no	no
$8k+6$	yes	complex	no	no
$8k+7$	no	real	no	no

Table 3: Existence of various types of spinors in Euclidean N dimensions.

2 Minkowski Space

We now consider representations of Spin($N-1$, 1).

2.1 Symmetry Property of General Representations

Here are a few remarks about the reality property of a representation of non-compact simple Lie algebra.

When given an irreducible representation matrix $\rho(T^a)$ for generators T^a, the "transpose representation" is defined by $-\rho(T^a)^T$ as in the compact case. However, there are potentially more representations related to this irreducible representation for the non-compact case. All together four sets of matrices

$$\rho(T^a), \qquad -\rho(T^a)^*, \qquad -\rho(T^a)^T, \qquad \rho(T^a)^\dagger \tag{43}$$

all satisfy the same Lie algebra. The first and the fourth are the same for the compact case, and so are the second and the third. We have already determined if $\rho(T^a)$ and $-\rho(T^a)^*$ give equivalent representations. We are yet to determine if $-\rho(T^a)^T$ give an equivalent representation.

2.2 Symmetry Property of Clifford Algebra

The existence and the symmetry properties of $\mathcal{C}$ and $\mathcal{T}$ matrices are the same in Euclidean N dimensions and Minkowski $(N-1)+1$ dimensions. This is because you can freely change the signature by multiplying i on gamma-matrices, which does not change the symmetry property of gamma-matrices. When either $\mathcal{C}$ or $\mathcal{T}$ exists, the generators $\frac{i}{4}[\Gamma^\mu, \Gamma^\nu]$ and their negative transpose $\frac{i}{4}[\Gamma^{\mu T}, \Gamma^{\nu T}]$ are unitarity equivalent. Since both of them exist for all even dimensions, and either one of them does for all odd dimensions, these two representations are always unitarity equivalent.

When $\mathcal{C}$ exists and is symmetric, one can go to a basis where $\mathcal{C}$ is an identity matrix and hence gamma-matrices are made all anti-symmetric. When $\mathcal{T}$ exists and is symmetric, one can go to a basis where $\mathcal{T}$ is an identity matrix and hence gamma-matrices are made all symmetric. In both cases, the generators can be represented by anti-symmetric matrices and hence $\rho(T^a) = -\rho(T^a)^T$. They are therefore manifestly identical for dimensions $N = 0, 1, 2, 6, 7 \bmod 8$. For $N = 3, 4, 5 \bmod 8$, they cannot be manifestly identical, but are still unitary equivalent.

As far as I know, the symmetry property of the Clifford algebra does not play any important roles.

2.3 Majorana Spinor

The Clifford Algebra is

$$\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}. \tag{44}$$

We use the metric $g^{\mu\nu} = \mathrm{diag}(-1, +1, \cdots, +1)$. With this metric, Γ^0 is anti-hermitian, while Γ^1 to Γ^{N-1} are all hermitian. Generators of rotations in Spin($N-1$,1) are given by

$$M^{\mu\nu} = \frac{i}{4}[\Gamma^\mu, \Gamma^\nu]. \tag{45}$$

It is easy to see that they satisfy the Lie algebra of SO($N-1$,1). Because only Γ^0 is anti-hermitian, they satisfy

$$\Gamma^0\Gamma^\mu\Gamma^0 = -\Gamma^{\mu\dagger}. \tag{46}$$

The important point is that you can start with the gamma matrices γ^i in $N-2$ dimensions, and tensor them with another Pauli matrix to obtain gamma matrices of Spin($N-1$, 1) without changing their reality properties. For instance, suppose you could make all gamma-matrices γ^1 to γ^{N-2} pure real in Euclidean $N-2$ dimensions. Then, you can define gamma-matrices in Minkowski $(N-1)+1$ dimensions by

$$\begin{aligned} \Gamma^0 &= 1 \otimes (i\sigma_2), && (47)\\ \Gamma^i &= \gamma^i \otimes \sigma_3, && (48)\\ \Gamma^{N-1} &= 1 \otimes \sigma_1. && (49) \end{aligned}$$

You find that all Γ's in Minkowski $(N-1)+1$ dimensions are also pure real. If γ^i are all pure imaginary, you can define pure imaginary Γ's

$$\begin{aligned} \Gamma^0 &= 1 \otimes (i\sigma_1), && (50)\\ \Gamma^i &= \gamma^i \otimes \sigma_3, && (51)\\ \Gamma^{N-1} &= 1 \otimes \sigma_2. && (52) \end{aligned}$$

Therefore, the question if you can impose Majorana condition is the same in Spin($N-2$) and Spin($N-1$,1).

Another way to see the existence of Majorana spinor is by the following analysis as what we have done in Euclidean case. First notice that

$$\Gamma^{\mu *} = (\Gamma^{\mu\dagger})^T = (\Gamma^0\Gamma^\mu\Gamma^0)^T = \begin{cases} \mathcal{C}\Gamma^0\Gamma^\mu\Gamma^0\mathcal{C}^{-1} = (\mathcal{C}\Gamma^0)\Gamma^\mu(\mathcal{C}\Gamma^0)^{-1} \\ \mathcal{T}\Gamma^0\Gamma^\mu\Gamma^0\mathcal{T}^{-1} = -(\mathcal{T}\Gamma^0)\Gamma^\mu(\mathcal{T}\Gamma^0)^{-1} \end{cases} \tag{53}$$

Therefore, you can make all gamma-matrices pure real when $\mathcal{C}\Gamma^0$ is symmetric, and pure imaginary when $\mathcal{C}\Gamma^0$ anti-symmetric. Note that $(\mathcal{C}\Gamma^0)^T = \Gamma^{0T}\mathcal{C}^T = -\mathcal{C}\Gamma^0\mathcal{C}^{-1}\mathcal{C}^T$, and hence the symmetry (anti-symmetry) of $\mathcal{C}$ implies anti-symmetry (symmetry) of $\mathcal{C}\Gamma^0$. Because $\mathcal{C}$ is anti-symmetric in $N = 2, 3, 4 \bmod 8$, these are indeed dimensions where Majorana spinors can exist. The Majorana condition can be written in the basis-independent manner, $\psi^* = \mathcal{C}\Gamma^0\psi$, or $\bar{\psi}^T = \Gamma^{0T}\psi^* = \Gamma^{0T}\mathcal{C}\Gamma^0\psi = -\mathcal{C}\Gamma^0\Gamma^0\psi = \mathcal{C}\psi$.

One obtains also the same result by considering the possible Majorana mass term in the action. Because of the Majorana condition $\bar{\psi}^T = \mathcal{C}\psi$, the mass term is

$$-m\int d^N\bar{\psi}\psi = -m\int d^N\psi^T\mathcal{C}^T\psi. \tag{54}$$

To be consistent with the anti-commuting nature of ψ, $\mathcal{C}^T$ and hence $\mathcal{C}$ must be anti-symmetric.

N	Weyl	reality	Majorana	Majorana–Weyl
$8k$	yes	complex	no	no
$8k+1$	no	real	no	no
$8k+2$	yes	real	yes	yes
$8k+3$	no	real	yes	no
$8k+4$	yes	complex	yes	no
$8k+5$	no	pseudo-real	no	no
$8k+6$	yes	pseudo-real	no	no
$8k+7$	no	pseudo-real	no	no

Table 4: Existence of various types of spinors in Minkowski $(N-1)+1$ dimensions.

3 Minkowski vs Euclidean Theory

In Minkowski space, you can impose Majorana condition on the fermion field operator and reduce the degrees of freedom by a half. However, after Wick rotation, you may not be able to impose the Majorana condition because the representation theory is different between Minkowski and Euclidean spaces. Is Dirac determinant defined in Wick-roated Euclidean path integral the right thing to look at?

To calculate the determinant of the Dirac operator, we first look at eigenequations

$$\gamma^i D_i \psi_n = i\lambda_n \psi_n. \tag{55}$$

$\lambda_n \in \mathbb{R}$ because the operator $\gamma^i D_i$ is anti-hermitian. In the presence of gauge field and space-time curvature, the covariant derivative is

$$D_i = \partial_i - igA_i^a T^a, \tag{56}$$

where T^a are representation matrices. Space-time curvature can also be included in this form by setting

$$gA_i^a T^a = \omega_i^{ab} \frac{\sigma^{ab}}{2}, \tag{57}$$

where ω_i is the spin connection and a, b the local Lorentz coordinates. Because we are interested in the question of Majorana spinors, we assume that the representation matrices T^a are all pure imaginary. Given an eigenfunction ψ_n, we can take complex conjugate of the eigenequation

$$-i\lambda_n \psi_n^* = \gamma^{i*} D_i \psi_n^* = -\mathcal{C}\gamma^i D_i \mathcal{C}^{-1} \psi_n^*. \tag{58}$$

In other words, $\mathcal{C}^{-1}\psi_n^*$ also has the eigenvalue $i\lambda_n$ for the Dirac operator. The question is if they are linearly independent.

If $\mathcal{C}$ is symmetric, you can go to the basis where $\mathcal{C} = 1$ and γ^i pure imaginary. Then the eigenequation Eq. (55) admits real solutions, and $\mathcal{C}^{-1}\psi_n^* = \psi_n$. Therefore, eigenfunctions ψ_n and $\mathcal{C}^{-1}\psi_n$ are not independent, and eigenvalues are not necessarily degenerate. On the other hand, if $\mathcal{C}$ is anti-symmetric, the eigenfunction is necessarily complex, and $\mathcal{C}^{-1}\psi_n^*$ gives a linearly independent eigenfunction. Then any eigenvalue $i\lambda_n$ is doubly degenerate. Therefore, one can naturally define a *square root* of the Dirac determinant, by choosing only one of the doubly degenerate eigenvalues. This result precisely corresponds to the situation in the Minkowski space where you can reduce the degrees of freedom by a half using the Majorana condition.

The Euclidean action for the Majorana fermion is

$$\int d^N x \frac{1}{2} [\psi^T C^{-1} (\gamma_i D_i - m) \psi]. \tag{59}$$

The path integral measure is the product of Grassman integration over all eigenmodes. If we focus on one eigenvalue $i\lambda_n$, the field can be expanded as

$$\psi = \psi_n \eta + C\psi_n^* \chi, \tag{60}$$

where η, χ are Grassman odd integration variables, while ψ_n and $C\psi_n^*$ are linearly independent eigenfunctions of $\gamma_i D_i$. Then the Euclidean action is

$$\int d^N x \psi^T C^{-1}(\gamma_i D_i - m)\psi = \int d^N x \frac{1}{2}(i\lambda_n - m)\eta\chi(\psi_n^T \psi_n^* + \psi_n^\dagger \psi_n) = (i\lambda_n - m)\eta\chi. \tag{61}$$

Grassman integration $\int d\eta d\chi$ gives just $(i\lambda_n - m)$. This way, you obtain a square root of the determinant.

In even dimensions, Eq. (55) can be split into two equations,

$$\gamma^i D_i \psi_{nL} = i\lambda_n \psi_{nR}, \tag{62}$$

$$\gamma^i D_i \psi_{nR} = i\lambda_n \psi_{nL}. \tag{63}$$

Therefore, the Dirac operator has the form

$$\begin{pmatrix} 0 & i\lambda_n \\ i\lambda_n & 0 \end{pmatrix}, \tag{64}$$

and hence has both eigenvalues $\pm i\lambda_n$. Keeping only one sign naturaly defines *another* square root of Dirac determinant. This corresponds to the Weyl spinor in Minkowski space.

The Euclidean action in this case is given by

$$\int d^N x[\bar{\psi}_R \gamma_i D_i \psi_L]. \tag{65}$$

The Grassman variables ψ_L and $\bar{\psi}_R$ are independent, and their integration yields $i\lambda_n$ for each eigenmodes. Actually, the relative phase between ψ_L and $\bar{\psi}_R$ can be changed and hence the integration can yield $i\lambda_n$ times a phase factor. This leaves the ambiguity in defining the determinant up to a phase, and the determinant is regarded as a section of a line bundle (determinant line bundle) over the space of gauge connections.

Finally, the question is if you can naturally take a *fourth root* of Dirac determinant by combining both of them. This is possible if the Euclidean Majorana action Eq. (59) splits into ψ_L and ψ_R. In $N = 8k + 2$ dimensions, C anti-commutes with γ^{8k+3}, and

$$\psi^T C^{-1}(\gamma_i D_i - m)\psi = \psi_L^T C^{-1}\gamma_i D_i \psi_L + \psi_R^T C^{-1}\gamma_i D_i \psi_R \tag{66}$$

Therefore, you can throw away ψ_R from path integral and obtain a fourth root of the determinant. On the other hand in $N = 8k + 4$ dimensions, C commutes with γ^{8k+5} and

$$\psi^T C^{-1}(\gamma_i D_i - m)\psi = \psi_R^T C^{-1}\gamma_i D_i \psi_L + \psi_L^T C^{-1}\gamma_i D_i \psi_R. \tag{67}$$

There is no consistent way to throw away either ψ_L or ψ_R from the path integral.

4 Supersymmetry

In $4k+2$ Minkowski dimensions, the are two inequivalent self-conjugate (real or pseudo-real) spinor representations. When you introduce supercharges, you can introduce any of them in either representation, and talk about $(1,0)$, $(1,1)$, $(2,0)$ supersymetry etc. In all other dimensions, however, one can only talk about $N=1$, $N=2$, etc supersymmetry. In $4k$ dimensions, the supercharges are complex, and once you introduce Q in one of the complex representations, their hermitian conjugate $\bar{Q}$ appear in the (inequivalent) conjugate representation and you need equal number of them. In odd dimensions, there is no Weyl representation and hence only one spinor representation.