# Notes on Dirac Operator, Chiral Anomalies, and Topology of the Gauge Field

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## 1 Dirac Index

The Dirac operator $i\not{D}$ on a compact closed Euclidean Space of even dimensions $D = 2n$ has the property that it anti-commutes

$$i\not{D}\gamma_{D+1} + \gamma_{D+1} i\not{D} = 0 \tag{1}$$

with

$$\gamma_{D+1} = (-i)^n \gamma^1 \cdots \gamma^D = i^n \gamma^D \cdots \gamma^1. \tag{2}$$

The Dirac operator is a hermitean operator. Throughout this note, I use the notation

$$D_\mu = \partial_\mu - iA_\mu = \partial_\mu - iA_\mu^a t^a, \tag{3}$$

and the gauge coupling constant appears only in the normalization of the gauge kinetic term

$$\mathcal{L} \ni -\frac{1}{2g^2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu}. \tag{4}$$

Just to fix the notation,

$$-iF_{\mu\nu} = -iF_{\mu\nu}^a t^a = [D_\mu, D_\nu]. \tag{5}$$

For later purposes, it is useful to introduce the differential forms

$$A = -iA_\mu dx^\mu = -iA_\mu^a t^a dx^\mu \tag{6}$$

and

$$F = -i\frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = -i\frac{1}{2} F_{\mu\nu}^a t^a dx^\mu \wedge dx^\nu. \tag{7}$$

Then $D = d + A$ and $F = dA + A^2$.

Because $(i\not{D})^2$ commutes with $\gamma_5$, we can look for their simultaneous eigenstates,

$$(i\not{D})^2 \psi_{n,\pm} = \lambda_n^2 \psi_{n,\pm}, \qquad \gamma_{D+1}\psi_{n,\pm} = \pm\psi_{n,\pm}. \tag{8}$$

Note that $\lambda_n^2$ is positive semi-definite because they are eigenvalues of a squared hermitean operator. We choose $\lambda_n \geq 0$ without a loss of generality. We normalize the eigenfunctions such that

$$\int d^D x \psi_{n,R}^\dagger \psi_{m,R} = \delta_{n,m}, \quad \int d^D x \psi_{n,L}^\dagger \psi_{m,L} = \delta_{n,m}, \quad \int d^D x \psi_{n,\pm}^\dagger \psi_{m,\mp} = 0. \tag{9}$$

Here, $R$ ($L$) refers to $\gamma_{D+1} = +1$ ($-1$) and we use both notations interchangingly.

Because of the anti-commutation, $i\not{D}\psi_{n,L}$ is right-handed, and $i\not{D}\psi_{n,R}$ is left-handed. It is easy to see that

$$(i\not{D})^2 i\not{D}\psi_{n,\pm} = i\not{D}(i\not{D})^2\psi_{n,\pm} = \lambda_n^2 i\not{D}\psi_{n,\pm}, \tag{10}$$

and hence $i\not{D}\psi_{n,\pm}$ has the same eigenvalue $(i\not{D})^2 = \lambda_n^2$ but has the opposite chirality. Hence,*

$$i\not{D}\psi_{n,L} = e^{i\phi_n}\lambda_n\psi_{n,R}, \qquad i\not{D}\psi_{n,R} = e^{-i\phi_n}\lambda_n\psi_{n,L}. \tag{11}$$

Therefore, the same eigenvalue $\lambda_n^2$ is shared between two chiralities, except for when the eigenfunction is annihilated by $i\not{D}$. Namely that the zero modes may not be paired, while non-zero modes are always paired. The eigenvalues of the Dirac opearator (without the square) are $\pm\lambda_n$ given by the eigenfunctions $\psi_{n,L} \pm e^{i\phi_n}\psi_{n,R}$:

$$i\not{D}(\psi_{n,L} \pm e^{i\phi_n}\psi_{n,R}) = e^{i\phi_n}\lambda_n\psi_{n,R} \pm e^{i\phi_n}e^{-i\phi_n}\lambda_n\psi_{n,L} = \pm\lambda_n(\psi_{n,L} \pm e^{i\phi_n}\psi_{n,R}). \tag{12}$$

As you vary the gauge field continuously, paired eigenstates may accidentally come down to zero. Yet the *difference* in the number of zero modes beween two chiralities does not change by this accidental pair of zero mode. We define the Dirac index

$$\text{index } i\not{D} = n_R^0 - n_L^0 \tag{13}$$

where $n_L^0$ ($n_R^0$) is the number of left-handed (right-handed) zero modes. The index does not change by a continuous variation of the gauge field and hence

*The relative phase can be removed by choosing an appropriate basis for eigenfunctions for a particular background gauge field, but there may be a reason why it may not be possible for a continuous variation of the gauge field. We will come back to this question when we discuss the gauge anomalies.

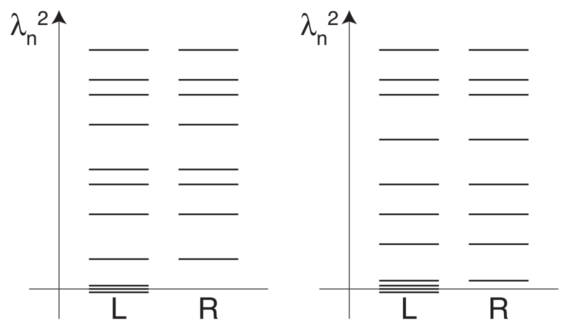


Figure 1: The structure of the eigenvalues of the Dirac operator. The non-zero modes are all paired between two chiralities, while the zero modes do not have to be paired. A pair may accidentally come to zero under continuous variation of the background gauge field as shown on the right, but the index does not change and is hence a topological invariant.

is a topological invariant. For the later purposes, we also write the index as

$$\begin{aligned} \text{index } i\not{D} &= \text{Tr } \gamma_{D+1} e^{-(i\not{D})^2/M^2} \\ &\equiv \sum_{n,\pm} \int d^D x \psi^\dagger_{n,\pm}(x) \gamma_{D+1} e^{-(i\not{D})^2/M^2} \psi_{n,\pm}(x) = \sum_{n,\pm} \pm e^{-\lambda_n^2/M^2}. \end{aligned} \quad (14)$$

Here, Tr refers to the trace over the space of all eigenfunctions as well as the Dirac and/or gauge indices. We use the notation tr for the trace over the Dirac and/or gauge indices alone. In this expression, the eigenmodes contribute with the opposite signs for the left-handed and right-handed chiralities due to the factor of $\gamma_{D+1}$, and they all cancel for non-zero modes because of the pairing. For the zero modes, however, the index remains. Note that the result is independent of the regular mass $M$.

The book shows in Section 19.2 how to compute this quantity in the limit $M \to \infty$. Eq. (19.74) holds in any dimensions,

$$(i\not{D})^2 = -\gamma^\mu \gamma^\nu D_\mu D_\nu = -\frac{1}{2}(g^{\mu\nu} + [\gamma^\mu, \gamma^\nu]) D_\mu D_\nu = -D^2 + \frac{1}{2}\sigma^{\mu\nu} F_{\mu\nu} \quad (15)$$

with $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Repeating the same calcuation in arbitrary even dimensions $D = 2n$, the equivalent of Eq. (19.75) is

$$\lim_{M\to\infty} \text{Tr} \gamma_{D+1} e^{-(i\not{D})^2/M^2}$$

$$= \int d^D x \lim_{M\to\infty} \mathrm{tr}\left[\gamma_{D+1}\frac{1}{n!}\left(\frac{1}{2M^2}\sigma^{\mu\nu}F_{\mu\nu}\right)^n\right]\langle x|e^{\partial^2/M^2}|x\rangle. \qquad (16)$$

The latter factor is

$$\langle x|e^{\partial^2/M^2}|x\rangle = \int \frac{d^D k}{(2\pi)^D} e^{-k^2/M^2} = \frac{M^D}{(4\pi)^n}\ . \qquad (17)$$

For the first factor, we find

$$\begin{aligned}
&\mathrm{tr}\left[\gamma_{D+1}\frac{1}{n!}\left(\frac{1}{2M^2}\sigma^{\mu\nu}F_{\mu\nu}\right)^n\right] \\
&\quad = \frac{1}{n!}\frac{1}{2^n M^D}\mathrm{tr}[\gamma_{D+1}\sigma^{\mu_1\mu_2}\cdots\sigma^{\mu_{D-1}\mu_D}]\mathrm{tr}[F_{\mu_1\mu_2}\cdots F_{\mu_{D-1}\mu_D}] \\
&\quad = \frac{1}{n!}\frac{1}{M^D}(-1)^n\epsilon^{\mu_1\mu_2\cdots\mu_{D-1}\mu_D}\mathrm{tr}[F_{\mu_1\mu_2}\cdots F_{\mu_{D-1}\mu_D}]. \qquad (18)
\end{aligned}$$

Putting them together,

$$\begin{aligned}
\text{index } i\not{D} &= \lim_{M\to\infty}\mathrm{Tr}\gamma_{D+1}e^{-(i\not{D})^2/M^2} \\
&= \int d^D x \frac{1}{n!}\frac{(-1)^n}{(4\pi)^n}\epsilon^{\mu_1\mu_2\cdots\mu_{D-1}\mu_D}\mathrm{tr}[F_{\mu_1\mu_2}\cdots F_{\mu_{D-1}\mu_D}] \\
&= \frac{1}{n!}\frac{(-1)^n i^n}{(2\pi)^n}\int \mathrm{tr}[F\wedge\cdots\wedge F] = \frac{1}{n!}\frac{(-i)^n}{(2\pi)^n}\int \mathrm{tr}F^n. \qquad (19)
\end{aligned}$$

The last expression is an integral of the Chern character

$$\mathrm{ch}(F) = \mathrm{tr}e^{-iF/2\pi}, \qquad (20)$$

picking the piece appropriate for the given dimension. This result is a special case of the general theorem called the Atiyah–Singer index theorem, which states that the analytical index (index $i\not{D}$ in this case) is the same as the topological index (the integral of the the Chern character in this case).

# 2 Topological Index

The topological index

$$\frac{1}{n!}\frac{(-i)^n}{(2\pi)^n}\int_M \mathrm{tr}F^n \qquad (21)$$

is an integer and it is a topological invariant as the name suggests. What it means is that it does not change under the continuous variation of the gauge field. Knowing this result, the reason is obvious; the analytical index is by definition an integer and does not change under the continuous variation of the gauge field. But how do we see it on the topological index directly?

One important point is that the Chern class does not depend on the metric of the manifold, and therefore does not rely on the differential structure (how jagged the sphere is, how much it is elongated, etc). Another important point is that it is a total derivative and depends only on the boundary condition (if on open space). We find, for example,

$$\mathrm{tr}F^2 = d\omega_3(A) = d\mathrm{tr}\left(AdA + \frac{2}{3}A^3\right), \tag{22}$$

$$\mathrm{tr}F^3 = d\omega_5(A) = d\mathrm{tr}\left(A(dA)^2 + \frac{3}{2}A^3dA + \frac{3}{5}A^5\right). \tag{23}$$

These $\omega_{2n+1}$'s are caled Chern-Simon terms.[†] They appear, for example, in the study of quantum Hall effects and three-dimensional QED with massive fermions.

We would like to see what the topological index is on a compact manifold such as a four-dimensional sphere $S^4$. For this purpose, we cover the $S^4$ with two patches, one covering the northern hemisphere and the other the southern one, and they overlap along the equator which is nothing but a three-dimensional sphere $S^3$. In general, when two patches are glued together, you need to perform a gauge transformation to connect them because they may be described in two different gauges. Such a gauge transformation is called the *transition function* in the theory of fibre bundles, a terminology in mathematics that basically refers to a general gauge theory. Our gauge field is their *connection,* our field strength is their *curvature.* The integral of $\mathrm{tr}F^2$ on the northern hemisphere is given by the Chern-Simon term on its boundary, namely on the equator $S^3$. Similarly the integral on the southern hemisphere is also given by the Chern-Simon term on $S^3$, but with the

[†]For general $n$, you can use the trick called "descent equations" invented our own Bruno Zumino, discussed in his Les Houches 1983 lectures, and also in Bruno Zumino, Wu Yong-Shi, and A. Zee, *Nucl. Phys.* **B 239**, 477 (1984), to work out explicit expressions for the Chern–Simon terms.

opposite orientation. Therefore,

$$\begin{aligned}\int_{S^4} \frac{-1}{8\pi^2}\mathrm{tr}F^2 &= \int_{S^4_{\mathrm{north}}} \frac{-1}{8\pi^2}\mathrm{tr}F^2 + \int_{S^4_{\mathrm{south}}} \frac{-1}{8\pi^2}\mathrm{tr}F^2 \\ &= \int_{S^3} \frac{-1}{8\pi^2}\omega_3(A_{\mathrm{north}}) - \int_{S^3} \frac{-1}{8\pi^2}\omega_3(A_{\mathrm{south}}). \end{aligned} \tag{24}$$

Because $A_{\mathrm{south}}$ is related to $A_{\mathrm{north}}$ by a gauge transformation

$$A_{\mathrm{south}} = g^{-1}dg + g^{-1}A_{\mathrm{north}}g, \tag{25}$$

we only need to know

$$\int_{S^4} \frac{-1}{8\pi^2}\mathrm{tr}F^2 = \int_{S^3} \frac{-1}{8\pi^2}\left[\omega_3(A_{\mathrm{north}}) - \omega_3(g^{-1}dg + g^{-1}A_{\mathrm{north}}g)\right]. \tag{26}$$

We will show below that it is given simply by

$$= \frac{1}{24\pi^2}\int_{S^3} \mathrm{tr}(g^{-1}dg)^3, \tag{27}$$

and hence it depends only on the gauge transformation on the equator. We will also see later that this is in fact an integer.

The difference between two $\omega_3$'s on the equator is easy to work out. From this point on, we use the notation $A = A_{\mathrm{north}}$, $A^g = A_{\mathrm{south}} = g^{-1}dg + g^{-1}Ag$.‡ We first work out

$$\begin{aligned} dA^g &= d(g^{-1}dg + g^{-1}Ag) \\ &= -(g^{-1}dg)^2 - g^{-1}dgg^{-1}Ag + g^{-1}dAg - g^{-1}Agg^{-1}dg \\ &= -(g^{-1}dg)^2 - \{g^{-1}Ag, g^{-1}dg\} + g^{-1}dAg. \end{aligned} \tag{28}$$

Note that we used $dg^{-1} = -g^{-1}dgg^{-1}$ which can be proven using $0 = d(g^{-1}g) = (dg^{-1})g + g^{-1}dg$. Now using the definition of the Chern–Simon term,

$$\begin{aligned} \omega_3(A^g) &= \mathrm{tr}\left(A^g dA^g + \frac{2}{3}A^{g3}\right) = \mathrm{tr}\left(A^g F^g - \frac{1}{3}A^{g3}\right) \\ &= \mathrm{tr}\left((g^{-1}dg + g^{-1}Ag)g^{-1}Fg - \frac{1}{3}(g^{-1}dg + g^{-1}Ag)^3\right) \end{aligned}$$

‡I'm a northern hemisphere supremacist. Sorry, Aussies!

$$
\begin{aligned}
&= \mathrm{tr}\left(dgg^{-1}F + AF - \frac{1}{3}(g^{-1}dg)^3 - (dgg^{-1})^2 A - dgg^{-1}A^2 - \frac{1}{3}A^3\right) \\
&= \omega_3(A) + \mathrm{tr}\left(dgg^{-1}F - \frac{1}{3}(g^{-1}dg)^3 + dgdg^{-1}A - dgg^{-1}A^2\right) \\
&= \omega_3(A) - \frac{1}{3}\mathrm{tr}(g^{-1}dg)^3 + \mathrm{tr}\left(dgg^{-1}dA + dgdg^{-1}A\right) \\
&= \omega_3(A) - \frac{1}{3}\mathrm{tr}(g^{-1}dg)^3 - d\mathrm{tr}\left(dgg^{-1}A\right). \qquad (29)
\end{aligned}
$$

Note that the latter two terms are closed. This is obvious for the last term because it is exact. For the middle term,

$$
\begin{aligned}
d\mathrm{tr}(g^{-1}dg)^3 &= -d\mathrm{tr} g^{-1}dgdg^{-1}dg \\
&= -\mathrm{tr} dg^{-1}dgdg^{-1}dg = -\mathrm{tr}(g^{-1}dg)^4 = +\mathrm{tr}(g^{-1}dg)^4 = 0. \qquad (30)
\end{aligned}
$$

In the last step, we used the cyclic property of the trace and the anti-commutation among odd-forms to show that it vanishes. The fact that the difference $\omega_3(A^g) - \omega_3(A)$ is closed is expected because of the gauge invariance of $\mathrm{tr}F^2$,

$$
d\omega_3(A^g) = \mathrm{tr}(F^g)^2 = \mathrm{tr}(g^{-1}Fg)^2 = \mathrm{tr}F^2 = d\omega_3(A). \qquad (31)
$$

The exact piece $d\mathrm{tr}(dgg^{-1}A)$ vanishes upon integration on $S^3$ because $S^3$ is closed. On the other hand,

$$
N = \frac{1}{24\pi^2}\int_{S^3} \mathrm{tr}(g^{-1}dg)^3 \qquad (32)
$$

is the *winding number* of the map from $S^3$ to the group.

Any simple group contains an $SU(2)$ subgroup. Topologically, $SU(2)$ is nothing but a three-dimensional sphere $S^3$. This can be seen easily as follows. Any $SU(2)$ matrix can be written as

$$
g(a) = a_0 + i\vec{\sigma}\cdot\vec{a}, \qquad (33)
$$

where the determinant is $\det g(a) = a_0^2 + \vec{a}^2 = 1$. The unitarity is easy to verify with this expression. Therefore, the space of all $SU(2)$ matrices can be parameterized by a four-dimensional unit vector $(a_0, \vec{a})$, nothing but $S^3$. Just like a rubber band $S^1$ can wrap around a water bottle, an $S^3$ (the equator) can wrap around the $S^3$ (the group) and you can count the number of times it wraps around. Clearly such a number is a topological number, independent of details (how much the rubber band is stretched, if the stretch is non-uniform, etc). Indeed, using the obvious map $(a_0, \vec{a}) = (\cos\theta, \sin\theta\sin\chi\cos\phi, \sin\theta\sin\chi\sin\phi, \sin\theta\cos\chi)$ where $\theta$, $\chi$, and $\phi$ are the coordinates of $S^3$, we have

$$g_1 = \cos\theta + i\sin\theta \begin{pmatrix} \cos\chi & \sin\chi e^{-i\phi} \\ \sin\chi e^{i\phi} & -\cos\chi \end{pmatrix}. \quad (34)$$

We can easily work out the integral and find

$$\begin{aligned} N &= \frac{1}{24\pi^2}\int_{S^3} \mathrm{tr}(g_1^{-1}dg_1)^3 = \frac{1}{24\pi^2}\int_{S^3} 12\sin^2\theta\sin\chi d\theta\wedge d\chi\wedge d\phi \\ &= \frac{1}{2\pi^2}\int_0^\pi \sin^2\theta d\theta \int_0^\pi \sin\chi d\chi \int_0^{2\pi} d\phi = 1. \end{aligned} \quad (35)$$

If we had used instead $g_n = g_1^n$, we find $N = n$.

In fact, mathematicians classify continuous maps from $S^n$ to a manifold $M$ using the homotopy group $\pi_n(M)$. If two maps can be continuously deformed to each other, you identify them. Each element of $\pi_n(M)$ refers to a class of maps under this identification. For any simple groups $G$, it is known that

$$\pi_3(G) = \mathbb{Z}. \quad (36)$$

Namely any maps from $S^3$ to a simple group $G$ can be classified according to the winding number. A representative map is given by the above $g_n$ embedded into $SU(2) \subset G$.

This way, the topological index determines the topology of the fibre bundle, namely it classifies how non-trivial the gauge field is.

# 3 Chiral Anomaly

Classically, massless Dirac Lagrangians in even dimensions coupled to the gauge field are invariant under the global chiral rotation $\psi' = e^{i\alpha\gamma_{D+1}}\psi$. This is because

$$\bar{\psi} i\not{D}\psi \to \psi^\dagger e^{-i\alpha\gamma_{D+1}}\gamma^0 i\not{D} e^{i\alpha\gamma_{D+1}}\psi = \psi^\dagger\gamma^0 i\not{D}\psi = \bar{\psi} i\not{D}\psi. \quad (37)$$

Correspondingly, the axial current is classically conserved,

$$\partial_\mu j_A^\mu = \partial_\mu(\bar{\psi}\gamma^\mu\gamma_{D+1}\psi) = 0. \quad (38)$$

To derive the Ward identity for the axial current, we can use the path integral with a local axial transformation

$$\psi'(x) = e^{i\alpha(x)\gamma_{D+1}}\psi(x). \tag{39}$$

Under this transformation, the Lagrangian density does change,

$$\bar{\psi}' i\not{D}\psi' = \bar{\psi} e^{i\alpha\gamma_{D+1}} i\not{D} e^{i\alpha\gamma_{D+1}}\psi = \bar{\psi} i\gamma^\mu \partial_\mu (i\alpha)\gamma_{D+1}\psi = -(\partial_\mu\alpha)\bar{\psi}\gamma^\mu\gamma_{D+1}\psi. \tag{40}$$

In the path integral, the fermion field is a dummy integration variable and we can replace $\psi$ by $\psi'$,

$$\begin{aligned}
&\int \mathcal{D}\psi\mathcal{D}\bar{\psi} e^{i\int d^D x \bar{\psi} i\not{D}\psi} \\
&\quad = \int \mathcal{D}\psi'\mathcal{D}\bar{\psi}' \exp\left[i\int d^D x \bar{\psi}' i\not{D}\psi'\right] \\
&\quad = \int \mathcal{D}\psi'\mathcal{D}\bar{\psi}' \exp\left[i\int d^D x (\bar{\psi} i\not{D}\psi - (\partial_\mu\alpha) j_A^\mu)\right] \\
&\quad = \int \mathcal{D}\psi'\mathcal{D}\bar{\psi}' e^{i\int d^D x \bar{\psi} i\not{D}\psi}\left[1 - i\int d^D x (\partial_\mu\alpha) j_A^\mu + O(\alpha)^2\right].
\end{aligned} \tag{41}$$

If we assume that the measure does not change under the change of variable $\mathcal{D}\psi'\mathcal{D}\bar{\psi}' = \mathcal{D}\psi\mathcal{D}\bar{\psi}$, we conclude

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi} e^{i\int d^D x \bar{\psi} i\not{D}\psi} \partial_\mu j_A^\mu = 0 \tag{42}$$

after an integration by parts because $\alpha(x)$ is arbitrary.

The question is if the measure is indeed invariant under the chiral rotation. Naively it is, because

$$\mathcal{D}\psi' = \mathcal{D} e^{i\alpha\gamma_{D+1}}\psi = \mathcal{D}\psi \left[\det e^{i\alpha\gamma_{D+1}}\right]^{-1} = \mathcal{D}\psi e^{-i\mathrm{Tr}\alpha(x)\gamma_{D+1}}. \tag{43}$$

Note that the Jacobian for Grassmann integral is the opposite from the ordinary ones. In the end, the exponent of the Jacobian is proportional to $\mathrm{tr}\gamma_{D+1} = 0$, and hence the Jacobian is unity. However, there is a subtlety. What multiplies $\mathrm{tr}\gamma_{D+1}$ is $\mathrm{Tr}\alpha(x) = \int d^D x \langle x|\alpha(x)|x\rangle = \int d^D x \alpha(x)\delta(x-x) = \infty$. Therefore, the exponent is actually $\infty \times 0$, and we have to carefully evaluate it.

For this purpose, we switch to the Euclidean path integral. One point you have to be careful about in the Euclidean case is that we do not regard $\bar{\psi} = \psi^\dagger \gamma^0$; we simply deal with $\psi$ and $\bar{\psi}$ as independent integration variable. This is because the Euclidean rotation $SO(D)$ is a compact group and is represented by unitary matrices on the spinors. Therefore, we should not put $\gamma^0$ in the definition of $\bar{\psi}$; the $SO(D)$ transformation of $\bar{\psi}$ is the same as $\psi^\dagger$ so that the Lagrangian is $SO(D)$ invariant. On the other hand, the chiral rotations of $\bar{\psi}$ need to be kept the same way as in the Minkowski case $\psi \to e^{i\alpha\gamma_{D+1}}\psi$ and $\bar{\psi} \to \bar{\psi} e^{i\alpha\gamma_{D+1}}$ so that the classical Lagrangian is invariant under the global chiral rotations. These two requirements are incompatible if we try to identify $\bar{\psi} = \psi^\dagger$ or $\psi^\dagger \gamma^0$.

The classical action splits into two chiralities,

$$\bar{\psi} i \not{D} \psi = \bar{\psi}_L i \not{D} \psi_R + \bar{\psi}_R i \not{D} \psi_L. \tag{44}$$

The chiral rotation is

$$\psi'_R = e^{i\alpha}\psi_R, \quad \psi'_L = e^{-i\alpha}\psi_L, \quad \bar{\psi}'_R = e^{i\alpha}\bar{\psi}_R, \quad \bar{\psi}'_L = e^{-i\alpha}\bar{\psi}_L, \tag{45}$$

so that the classical Lagrangian is invariant for a constant $\alpha$. Using the eigenmodes of $(i\not{D})^2$ we discussed in the earlier section, we can expand the field

$$\begin{aligned} \psi_R &= \sum_n \psi_{n,R} a_n, \qquad & \psi_L &= \sum_n \psi_{n,L} b_n, \\ \bar{\psi}_R &= \sum_n \psi^\dagger_{n,R} \bar{a}_n, \qquad & \bar{\psi}_L &= \sum_n \psi^\dagger_{n,L} \bar{b}_n. \end{aligned} \tag{46}$$

Note again $\bar{a}_n \neq (a_n)^\dagger$. Assume for definiteness that index $i\not{D} = k > 0$, so that we have $n$ zero modes for $\psi_R$ and $\bar{\psi}_R$, and no zero modes for $\psi_L$ and $\bar{\psi}_L$. Then the path integral measure is simply

$$\mathcal{D}\psi_R \mathcal{D}\psi_L \mathcal{D}\bar{\psi}_R \mathcal{D}\bar{\psi}_L = \left( \prod_{n\neq 0} da_n db_n d\bar{a}_n d\bar{b}_n \right) da_0^1 \cdots da_0^k d\bar{a}_0^1 \cdots d\bar{a}_0^k. \tag{47}$$

Under the global chiral rotation $\psi \to e^{i\alpha\gamma_{D+1}}\psi$, $a_n$ and $b_n$ pick up the opposite phases and the Jacobians cancel for each pair. However the zero modes are not paired because of the non-zero index and hence

$$\mathcal{D}\psi'_L \mathcal{D}\psi'_R \mathcal{D}\bar{\psi}'_L \mathcal{D}\bar{\psi}'_R = \mathcal{D}\psi_L \mathcal{D}\psi_R \mathcal{D}\bar{\psi}_L \mathcal{D}\bar{\psi}_R e^{-2ik\alpha}. \tag{48}$$

Therefore, the measure is *not* invariant due to the mismatch between the left- and right-handed chirality modes. Because the cancellation of Jacobians between left and right is guaranteed for non-zero modes by the pairing, this result is *exact* for a global chiral rotation.

For local chiral rotations, what we need is the exponent of the Jacobian

$$\lim_{M\to\infty} \mathrm{Tr}\alpha(x)\gamma_{D+1}e^{-(i\not{D})^2/M^2}. \tag{49}$$

Even though we computed this quantity for a constant $\alpha$ earlier, the computation did not depend on the actual form of $\alpha(x)$ in the $M \to \infty$ limit. Therefore, we find

$$\lim_{M\infty} \mathrm{Tr}\alpha(x)\gamma_{D+1} = \frac{1}{n!}\frac{(-i)^n}{(2\pi)^n}\int \alpha(x)\mathrm{tr}F^n. \tag{50}$$

Using this result, the measure changes as

$$\mathcal{D}\psi'\mathcal{D}\bar{\psi}' = \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left[-2i\frac{1}{n!}\frac{(-i)^n}{(2\pi)^n}\int \alpha(x)\mathrm{tr}F^n\right]. \tag{51}$$

Note that the case for a global chiral rotation is given precisely by the topological index, reducing back to Eq. (**??**).

Therefore the Ward identity now includes the anomalous violation of chiral symmetry

$$\begin{aligned}
&\int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{i\int d^Dx\bar{\psi}i\not{D}\psi} \\
&= \int \mathcal{D}\psi'\mathcal{D}\bar{\psi}'e^{i\int d^Dx\bar{\psi}i\not{D}\psi}\left[1-\int d^Dx(\partial_\mu\alpha)j^\mu_A + O(\alpha)^2\right] \\
&= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\left[1-2i\frac{1}{n!}\frac{(-i)^n}{(2\pi)^n}\int \alpha(x)\mathrm{tr}F^n + O(\alpha^2)\right] \\
&\qquad e^{i\int d^Dx\bar{\psi}i\not{D}\psi}\left[1-i\int d^Dx(\partial_\mu\alpha)j^\mu_A + O(\alpha)^2\right],
\end{aligned} \tag{52}$$

and hence

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{i\int d^Dx\bar{\psi}i\not{D}\psi}\left[-2i\frac{1}{n!}\frac{(-i)^n}{(2\pi)^n}\int \alpha(x)\mathrm{tr}F^n - i\int d^Dx(\partial_\mu\alpha)j^\mu_A\right] = 0. \tag{53}$$

The current conservation is now violated,[§]

$$\partial_\mu j^\mu_A = 2\frac{1}{n!}\frac{(-1)^n}{(4\pi)^n}\epsilon^{\mu_1\mu_2\cdots\mu_{D-1}\mu_D}\mathrm{tr}(F_{\mu_1\mu_2}\cdots F_{\mu_{D-1}\mu_D}). \tag{54}$$

Because the normalization of the r.h.s. is fixed by its integral which is supposed to be an integer, there cannot be a renormalization of the r.h.s. due to higher orders in perturbation theory. It is believed, therefore, that this equation is *exact* to all orders in perturbation theory(Adler–Bardeen theorem). This point is important because we use this equation to compute the $\pi^0 \to \gamma\gamma$ decay amplitude which is subject to corrections due to the gluon exchange at low momenta in the triangle diagram.

We now interpret the result as a non-conservation of the chiral current in the Minkowski space. The zero mode in the Euclidean space corresponds to the level-crossing in the Minkowski space. Using the topological invariance, we can open up $S^n$ to $\mathbb{R}^n$ as long as the field strength dies sufficiently fast $F \to 0$ at the boundary so that $F^n$ remains integrable. We choose the gauge $A_D = 0$ along the direction $x^D$ which we rotate to the Minkowski time $\tau = x^D = it$. The zero mode equation is then

$$i\left[\gamma^D\partial_\tau + \gamma^i D_i\right]\psi_0 = 0. \tag{55}$$

Here, $i = 1, \cdots, D-1$. Let us assume that the gauge field $A_i$ has a very slow dependence on $\tau$ so that we can integrate this equation using the adiabatic approximation. Namely that we can use the instantaneous eigenvalues

$$\gamma^D\gamma^i D_i\psi_0 = E(\tau)\psi_0. \tag{56}$$

Note that $i\gamma^D\gamma^i$ is hermitean, as well as $-iD_i$. The zero-mode equation can then be written as

$$\partial_\tau\psi_0 + E(\tau)\psi_0 = 0, \tag{57}$$

and hence

$$\psi_0(\tau) = \psi_0(0)e^{-\int_0^\tau d\tau' E(\tau')}. \tag{58}$$

The only way the zero mode can be normalizable is when

$$\lim_{\tau\to\infty} E(\tau) > 0, \qquad \lim_{\tau\to-\infty} E(\tau) < 0, \tag{59}$$

[§]I haven't identified the source of the opposite sign from Eq. (19.80) in the book.

namely the level crosses zero. In the Minkowski space, it corresponds to an energy level that is originally in the Dirac sea as one of the negative energy solutions and hence is occupied but sticks above the Dirac sea in the future with a positive energy. Therefore the Dirac zero mode in the Euclidean space gives the appearance of a fermion and hence changes the fermion number for one of the chiralities. There is no corresponding zero mode for the opposite chirality in the Euclidean space because of the lack of pairing for the zero modes, but it simply corresponds to the non-normalizable mode whose energy goes from positive to negative instead. This is how the chiral current is not conserved, where the amount of non-conservation is given by twice the index.

# 4 Gauge Anomaly

# 5 Global Anomaly

# 6 Gravitational Anomaly