Renormalization of the Yukawa Theory

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We solve Problem 10.2 in Peskin–Schroeder. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - M)\psi - ig\bar{\psi}\gamma^5\psi\phi. \tag{1}$$

The action is invariant under the parity $\mathcal{P}^\dagger\psi(\vec{x},t)\mathcal{P} = \gamma^0\psi(-\vec{x},t)$, $\mathcal{P}^\dagger\phi(\vec{x},t)\mathcal{P} = -\phi(-\vec{x},t)$. The parity transformation of the Lagrangian density is (note that only fields are operators)

$$\begin{aligned}
&\mathcal{P}^\dagger\mathcal{L}(\vec{x},t)\mathcal{P} \\
&= \frac{1}{2}(\partial_\mu\mathcal{P}^\dagger\phi\mathcal{P})^2 - \frac{1}{2}m^2\mathcal{P}^\dagger\phi^2\mathcal{P} + \mathcal{P}^\dagger\bar{\psi}\mathcal{P}(i\partial\!\!\!/ - M)\mathcal{P}^\dagger\psi\mathcal{P} - ig\mathcal{P}^\dagger\bar{\psi}\mathcal{P}\gamma^5\mathcal{P}^\dagger\psi\mathcal{P}\mathcal{P}^\dagger\phi\mathcal{P} \\
&= \frac{1}{2}(-\partial_\mu\phi(-\vec{x},t))^2 - \frac{1}{2}m^2(-\phi(-\vec{x},t))^2 \\
&\quad +\bar{\psi}(-\vec{x},t)\gamma^0(i\partial\!\!\!/ - M)\gamma^0\psi(-\vec{x},t) - ig\bar{\psi}(-\vec{x},t)\gamma^0\gamma^5\gamma^0\psi(-\vec{x},t)(-\phi(-\vec{t},t)). \quad (2)
\end{aligned}$$

Now we use the properties of the gamma matrices $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$ and $\{\gamma^\mu,\gamma^5\} = 0$. Note also $\vec{\nabla}\phi(-\vec{x},t) = -(\vec{\nabla}\phi)(-\vec{x},t)$ etc. The minus signs appear twice in the first two terms and cancel. The third term is

$$\begin{aligned}
\bar{\psi}(-\vec{x},t)\gamma^0 i\gamma^\mu\partial_\mu\gamma^0\psi(-\vec{x},t) &= \bar{\psi}(-\vec{x},t)\gamma^0 i(\gamma^0\partial_0 + \gamma^i\partial_i)\gamma^0\psi(-\vec{x},t) \\
&= \bar{\psi}(-\vec{x},t) i(\gamma^0\partial_0 - \gamma^i\partial_i)\psi(-\vec{x},t) \\
&= \bar{\psi}(-\vec{x},t)[i(\gamma^0\partial_0 + \gamma^i\partial_i)\psi](-\vec{x},t). \quad (3)
\end{aligned}$$

The very last term is

$$\begin{aligned}
-ig\bar{\psi}(-\vec{x},t)\gamma^0\gamma^5\gamma^0\psi(-\vec{x},t)(-\phi(-\vec{t},t)) &= ig\bar{\psi}(-\vec{x},t)(-\gamma^5)\psi(-\vec{x},t)\phi(-\vec{t},t) \\
&= [-ig\bar{\psi}\gamma^5\psi\phi](-\vec{x},t). \quad (4)
\end{aligned}$$

Therefore, $\mathcal{P}^\dagger\mathcal{L}(\vec{x},t)\mathcal{P} = \mathcal{L}(-\vec{x},t)$, and hence $S = \int dt d\vec{x}\mathcal{L}$ is invariant.

1 (a)

The number of loop integrals is given by the number of independent four-momenta in a given diagram. Each propagator has its own four-momentum, while each vertex enforces four-momentum conservation. However, there always remains one overall four-momentum conservation of the diagram. Hence,

$$L = P_f + P_s - V_Y + 1, \tag{5}$$

where L is the number of loop integrals, $P_{f,s}$ the number of fermion or scalar propagators, respectively, and V_Y the number of Yukawa vertices. Since each Yukawa vertex comes with two fermion lines and one scalar line,

$$V_Y = \frac{1}{2}(2P_f + N_f) = 2P_s + N_s. \tag{6}$$

Here, $N_{f,s}$ are the number of external fermion or scalar lines, respectively. Each propagator is counted twice as it connects two vertices, while an external line connects to only one vertex. The superficial degrees of divergence is

$$D = 4L - P_f - P_s. \tag{7}$$

Eliminating V_Y, L, $P_{f,s}$ from the above equations, we find

$$L = 4 - N_s - \frac{3}{2}N_f. \tag{8}$$

This is the same result as in QED, even though the Yukawa theory does not benefit from the additional softening as in QED which is due to the gauge invariance of the theory.

Therefore the superficially divergent diagrams are: two-point function of scalar ($D = 2$), two-point function of fermion ($D = 1$), four-point function of scalar ($D = 0$), and fermion-fermion-scalar three-point function ($D = 0$). Note that the one-point function of scalar is superficially divergent ($D = 3$), but it vanishes identically because of the parity invariance. The same is true with the scalar three-point function ($D = 1$) which also vanishes identically.

Because the scalar four-point function is divergent, we actually need a counter term to cancel it, namely we need a counter term of the form $-\frac{\delta_\lambda}{4!}\phi^4$. In other words, without the interaction $-\frac{\lambda}{4!}\phi^4$ in the bare Lagrangian, the theory is not renormalizable.

All divergences are then taken care of by counter terms already present in the bare Lagrangian

$$\mathcal{L}_{ct} = \frac{1}{2}\delta_{Z_\phi}(\partial_\mu\phi)^2 - \frac{1}{2}\delta_m\phi^2 + \delta_{Z_\psi}\bar{\psi}i\not{\partial}\psi - \delta_M\bar{\psi}\psi - i\delta_g\bar{\psi}\gamma^5\psi\phi - \frac{1}{4!}\delta_\lambda\phi^4. \tag{9}$$

To fix all six counter terms, we need to impose the same number of renormalization conditions.

Together with the new vertex, the counting changes as follows:

$$\begin{aligned} L &= P_f + P_s - V_Y - V_\lambda + 1, & (10)\\ 2V_Y &= 2P_f + N_f, & (11)\\ V_Y + 4V_\lambda &= 2P_s + N_s, & (12)\\ D &= 4L - P_f - P_s. & (13) \end{aligned}$$

V_λ is the number of ϕ^4 vertices. Eliminating V_Y, V_λ, L, $P_{f,s}$, we find the same result

$$D = 4 - N_s - \frac{3}{2}N_f. \tag{14}$$

2 (b)

The problem encourages to identify a clever choice of the external momenta to simplify the calculations as the divergence structure does not depend on the choice of renomalization conditions. We choose zero-momentum subtraction, namely that corrections to the 1PI two-point functions vanish both for scalar and fermion, their derivatives also vanish at the zero momentum, and also the corrections to the amputated fermion-fermion-scalar three-point function and scalar four-point function vanish at the zero momentum. This choice considerably simply the computation of counter terms.

I have intentionally kept finite terms until the last step for each counter term. For the pupose of solving the problem alone, which is asking only for the pole terms, this is not necessary and one can simlify the calculations at much earlier stages. However, for computation of actual amplitudes, it is *necessary* to keep all terms that remain finite in the $\epsilon \to 0$ limit. After all, finite terms are what give you physical amplitudes.

2.1 Scalar Two-Point Function

The one-loop diagram of the scalar two-point function is

$$-iM^2(p^2) = -g^2 \int \frac{d^dk}{(2\pi)^d} \mathrm{Tr}\gamma^5 \frac{i}{\not{k} + \not{p} - M} \gamma^5 \frac{i}{\not{k} - M}. \tag{15}$$

Here, the overall minus sign is there because of the closed fermion loop. As usual, we first rewrite the fermion propagator as $1/(\not{k}-M) = (\not{k}+M)/(k^2-M^2)$, and combine two denominators using the Feynman parameter

$$-iM^2(p^2) = g^2 \int \frac{d^d k}{(2\pi)^d} \int_0^1 dz \frac{\mathrm{Tr}\gamma^5(\not{k}+\not{p}+M)\gamma^5(\not{k}+M)}{[z((k+p)^2-M^2)+(1-z)(k^2-M^2)]^2}. \tag{16}$$

The denominator can be simplified as

$$\begin{aligned} & z((k+p)^2-M^2)+(1-z)(k^2-M^2) \\ &= k^2+2zk\cdot p+zp^2-M^2 \\ &= (k+zp)^2+z(1-z)p^2-M^2. \end{aligned} \tag{17}$$

Shifting k to $k-zp$,

$$-iM^2(p^2) = g^2 \int \frac{d^d k}{(2\pi)^d} \int_0^1 dz \frac{\mathrm{Tr}\gamma^5(\not{k}+(1-z)\not{p}+M)\gamma^5(\not{k}-z\not{p}+M)}{[k^2+z(1-z)p^2-M^2]^2}. \tag{18}$$

Because the integration volume $d^d k$ and the denominator are even in k, the odd powers in k in the numerator yield vanshing integrals. Therefore, we need to only retain terms quadratic and zeroth order in k in the numerator. Using the anti-commutation relation and $\gamma^5\gamma^5 = 1$,

$$\begin{aligned} & \mathrm{Tr}\gamma^5(\not{k}+(1-z)\not{p}+M)\gamma^5(\not{k}-z\not{p}+M) \\ &= \mathrm{Tr}\gamma^5\gamma^5(-\not{k}-(1-z)\not{p}+M)(\not{k}-z\not{p}+M) \\ &= \mathrm{Tr}[-(\not{k})^2+z(1-z)(\not{p})^2+M^2] = 4[-k^2+z(1-z)p^2+M^2]. \end{aligned} \tag{19}$$

Therefore,[1]

$$\begin{aligned} -iM^2(p^2) &= g^2 \int \frac{d^d k}{(2\pi)^d} \int_0^1 dz \frac{4[-k^2+z(1-z)p^2+M^2]}{[k^2+z(1-z)p^2-M^2]^2} \\ &= 4g^2 \int \frac{d^d k}{(2\pi)^d} \int_0^1 dz \left[-\frac{1}{k^2+z(1-z)p^2-M^2} + \frac{2z(1-z)p^2}{[k^2+z(1-z)p^2-M^2]^2} \right] \\ &= 4g^2 \int_0^1 dz \left[-\frac{-i\Gamma(-1+\epsilon)}{(4\pi)^{2-\epsilon}} [M^2-z(1-z)p^2]^{1-\epsilon} \right. \\ & \qquad \left. + 2z(1-z)p^2 \frac{i\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} [M^2-z(1-z)p^2]^{-\epsilon} \right]. \end{aligned} \tag{20}$$

[1] Note that my notation is slightly different from Peskin-Schroeder: I take $d = 4-2\epsilon$ instead of their $d = 4-\epsilon$ which keeps the ϵ dependence simpler.

Together with the counter terms and the tree-level piece, the two-point function is

$$i\Gamma(p^2) = i(p^2 - m^2) - iM^2(p^2) + i(\delta_{Z_\phi} p^2 - \delta_m). \tag{21}$$

With the zero-momentum subtraction scheme, we require $\Gamma(0) = -m^2$, $d\Gamma/dp^2(0) = 1$. Therefore the renormalization conditions are

$$M^2(0) + \delta_m = 0, \tag{22}$$

$$\left.\frac{d^2}{dp^2} M^2\right|_{p^2=0} - \delta_{Z_\phi} = 0. \tag{23}$$

The mass counter term is therefore

$$\begin{aligned} \delta_m &= 4g^2 \int_0^1 dz \frac{\Gamma(-1+\epsilon)}{(4\pi)^{2-\epsilon}} [M^2]^{1-\epsilon} = 4g^2 \frac{\Gamma(-1+\epsilon)}{(4\pi)^{2-\epsilon}} (M^2)^{1-\epsilon} \\ &= -4g^2 \frac{1}{(4\pi)^2} \frac{1}{\epsilon} M^2 + \text{finite}. \end{aligned} \tag{24}$$

Here I used the Laurent expansion,

$$\Gamma(-1+\epsilon) = \frac{1}{-1+\epsilon}\Gamma(\epsilon) = -(1+\epsilon+O(\epsilon)^2)\left(\frac{1}{\epsilon} - \gamma + O(\epsilon)\right) = -\frac{1}{\epsilon} + \text{regular}. \tag{25}$$

The wavefunction renormalization is somewhat more involved, but is still straightforward

$$\begin{aligned} \delta_Z &= \left.\frac{d^2}{dp^2} M^2(p^2)\right|_{p^2=0} \\ &= 4g^2 \int_0^1 dz \left[-\frac{\Gamma(-1+\epsilon)}{(4\pi)^{2-\epsilon}} (1-\epsilon)(-z(1-z))[M^2]^{-\epsilon} \right. \\ &\qquad \left. - 2z(1-z)\frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} [M^2]^{-\epsilon} \right] \\ &= -4g^2 \int_0^1 dz 3z(1-z) \frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} (M^2)^{-\epsilon} \\ &= -2g^2 \frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} (M^2)^{-\epsilon} = -2g^2 \frac{1}{(4\pi)^2}\frac{1}{\epsilon} + \text{finite}. \end{aligned} \tag{26}$$

We used the identity $(-1+\epsilon)\Gamma(-1+\epsilon) = \Gamma(\epsilon)$ in the fourth line. Note that the pole at $\epsilon = 1$ indicates a quadratic divergence, while that at $\epsilon = 0$ a logarithmic divergence.

Even though this is not required in the problem, it is instructive to see that the sum of the one-loop diagram and the counter terms is now manifestly finite for all momenta. What the counter terms do is to subtract the integrand at $p^2 = 0$ and the piece at first order in p^2. Looking at the first term in the integral Eq. (20),

$$\begin{aligned}
&-\frac{-i\Gamma(-1+\epsilon)}{(4\pi)^{2-\epsilon}}\left\{[M^2 - z(1-z)p^2]^{1-\epsilon} - [M^2]^{1-\epsilon} - (1-\epsilon)z(1-z)p^2[M^2]^{-\epsilon}\right\} \\
&= \frac{i\Gamma(-1+\epsilon)}{(4\pi)^{2-\epsilon}}\left\{[M^2 - z(1-z)p^2](1-\epsilon\ln[M^2 - z(1-z)p^2])\right. \\
&\qquad \left. -M^2(1-\epsilon\ln M^2) + (1-\epsilon)z(1-z)p^2(1-\epsilon\ln M^2) + O(\epsilon)^2\right\} \\
&= \frac{i\Gamma(-1+\epsilon)}{(4\pi)^{2-\epsilon}}\left\{-\epsilon[M^2 - z(1-z)p^2]\ln[M^2 - z(1-z)p^2])\right. \\
&\qquad \left. +\epsilon M^2\ln M^2 - \epsilon z(1-z)p^2 - \epsilon z(1-z)p^2\ln M^2 + O(\epsilon)^2\right\} \\
&= \frac{-i\Gamma(-1+\epsilon)}{(4\pi)^{2-\epsilon}}\epsilon\left\{[M^2 - z(1-z)p^2]\ln\frac{M^2 - z(1-z)p^2}{M^2} + z(1-z)p^2 + O(\epsilon)\right\} \\
&= \frac{i}{(4\pi)^2}\left\{[M^2 - z(1-z)p^2]\ln\frac{M^2 - z(1-z)p^2}{M^2} + z(1-z)p^2\right\} + O(\epsilon)
\end{aligned} \tag{27}$$

which is manifestly finite. Also the second term is

$$\begin{aligned}
&2z(1-z)p^2\frac{i\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}}\left\{[M^2 - z(1-z)p^2]^{-\epsilon} - [M^2]^{-\epsilon}\right\} \\
&= 2z(1-z)p^2\frac{i\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}}\left\{-\epsilon\ln\frac{M^2 - z(1-z)p^2}{M^2} + O(\epsilon)^2\right\} \\
&= -2z(1-z)p^2\frac{i}{(4\pi)^2}\ln\frac{M^2 - z(1-z)p^2}{M^2} + O(\epsilon)
\end{aligned} \tag{28}$$

Therefore, the full two-point function

$$\begin{aligned}
&i\Gamma(p^2) = i(p^2 - m^2) - iM^2(p^2) + i(\delta_{Z_\phi}p^2 - \delta_m) \\
&= i(p^2 - m^2) \\
&\quad +4g^2\frac{i}{(4\pi)^2}\int_0^1 dz\left[[M^2 - 3z(1-z)p^2]\ln\frac{M^2 - z(1-z)p^2}{M^2} + z(1-z)p^2\right]
\end{aligned} \tag{29}$$

is manifestly finite. Note, however, that the location of the pole $\Gamma(p^2) = 0$ is no longer $p^2 = m^2$ but is shifted, and residue is not unity either. Therefore, the physical mass is corrected from m^2. This is because we employed the zero-momentum subtraction scheme instead of the on-shell scheme. The former is easier for the purpose of seeing that all physical amplitudes are finite, while the latter admits physical interpretation of results more readily.

The location of the pole can be identified by writing $m_{pole}^2 = m^2 + \delta m^2$ and require $\Gamma(m^2 + \delta m^2) = 0$. Note that δm^2 is $O(g^2)$, and hence one does not keep it inside the integral which is already $O(g^2)$. Therefore,

$$\begin{aligned} 0 &= i\Gamma(p^2 = m_{pole}^2) \\ &= i\delta m^2 + 4g^2 \frac{i}{(4\pi)^2} \int_0^1 dz \left[[M^2 - 3z(1-z)m^2] \ln \frac{M^2 - z(1-z)m^2}{M^2} + z(1-z)m^2 \right] \end{aligned} \tag{30}$$

The residue at the pole is $Z = (d\Gamma(m_{pole}^2)/dp^2)^{-1} \neq 1$ and therefore any N-point function needs to be multiplied by $Z^{-N/2}$ to work out the S-matrix elements according to the LSZ reduction formula. See Ch 7.2 in Peskin–Schroeder. Explicitly,

$$\begin{aligned} Z^{-1} &= \left. \frac{d\Gamma(p^2)}{dp^2} \right|_{p^2 = m_{pole}^2} \\ &= 1 + \frac{4g^2}{(4\pi)^2} \int_0^1 dz z(1-z) \left[-3 \ln \frac{M^2 - z(1-z)m_{pole}^2}{M^2} - \frac{M^2 - 3z(1-z)m_{pole}^2}{M^2 - z(1-z)m_{pole}^2} + 1 \right]. \end{aligned} \tag{31}$$

2.2 Fermion Two-Point Function

The one-loop diagram for the fermion two-point function is

$$\begin{aligned} -i\Sigma(\not{p}) &= g^2 \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - m^2} \gamma^5 \frac{i}{\not{k} + \not{p} - M} \gamma^5 \\ &= -g^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - m^2} \frac{-\not{k} - \not{p} + M}{(k+p)^2 - M^2} \\ &= -g^2 \int \frac{d^d k}{(2\pi)^d} \int_0^1 dz \frac{-\not{k} - \not{p} + M}{[k^2 + 2k \cdot p + zp^2 - (1-z)m^2 - zM^2]^2} \\ &= -g^2 \int \frac{d^d k}{(2\pi)^d} \int_0^1 dz \frac{-(1-z)\not{p} + M}{[k^2 + z(1-z)p^2 - (1-z)m^2 - zM^2]^2} \\ &= -g^2 \int_0^1 dz [-(1-z)\not{p} + M] \frac{i\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} [(1-z)m^2 + zM^2 - z(1-z)p^2]^{-\epsilon} \end{aligned} \tag{32}$$

The full 1PI two-point function at the one-loop contains the tree-level piece as well as counter terms,

$$i\Gamma(\not{p}) = i(\not{p} - M) - i\Sigma(\not{p}) + i(\delta_{Z_\psi} \not{p} - \delta_M). \tag{33}$$

We require that the corrections cancel for zero-momentum which fixes the counter terms,

$$\Sigma(0) + \delta_M = 0, \tag{34}$$

$$\frac{d}{d\not{p}}\Sigma(\not{p})\bigg|_{\not{p}=0} - \delta_{Z_\psi} = 0. \tag{35}$$

The mass counter term is

$$\begin{aligned}
\delta_M &= -\Sigma(0) \\
&= -g^2 \int_0^1 dz M \frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}}[(1-z)m^2 + zM^2]^{-\epsilon} \\
&= -g^2 M \frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} \frac{1}{1-\epsilon}\left[\frac{1}{M^2 - m^2}[z(M^2 - m^2) + m^2]^{1-\epsilon}\right]_0^1 \\
&= -g^2 M \frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} \frac{1}{1-\epsilon} \frac{(M^2)^{1-\epsilon} - (m^2)^{1-\epsilon}}{M^2 - m^2} \\
&= -g^2 M \frac{1}{(4\pi)^2}\frac{1}{\epsilon} + \text{finite}.
\end{aligned} \tag{36}$$

The wave function counter term is

$$\begin{aligned}
\delta_{Z_\psi} &= g^2 \frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} \times \\
&\frac{d}{d\not{p}}\left[\int_0^1 dz[-(1-z)\not{p} + M][(1-z)m^2 + zM^2 - z(1-z)p^2]^{-\epsilon}\right]\bigg|_{\not{p}=0}
\end{aligned} \tag{37}$$

In general, the derivative with respect to $\not{p}$ hits $p^2 = (\not{p})^2$. However, with the zero-momentum subtraction, we set $\not{p} = 0$ in the end and hence $d(p^2)/d\not{p} = 2\not{p}$ does not contribute. Therefore,

$$\delta_{Z_\psi} = g^2 \frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} \int_0^1 dz[-(1-z)][(1-z)m^2 + zM^2]^{-\epsilon} \tag{38}$$

The integral is easily done this way,

$$\begin{aligned}
&\int_0^1 dz(1-z)[(1-z)m^2 + zM^2]^{-\epsilon} \\
&= \frac{1}{1-\epsilon}\frac{\partial}{\partial m^2}\int_0^1 dz[(1-z)m^2 + zM^2]^{1-\epsilon}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{1-\epsilon}\frac{\partial}{\partial m^2}\left[\frac{1}{2-\epsilon}\frac{1}{M^2-m^2}[z(M^2-m^2)+m^2]^{2-\epsilon}\right]_0^1 \\
&= \frac{1}{1-\epsilon}\frac{\partial}{\partial m^2}\frac{1}{2-\epsilon}\frac{(M^2)^{2-\epsilon}-(m^2)^{2-\epsilon}}{M^2-m^2} \\
&= \frac{1}{(1-\epsilon)(2-\epsilon)}\frac{-(2-\epsilon)(m^2)^{1-\epsilon}(M^2-m^2)-((M^2)^{2-\epsilon}-(m^2)^{2-\epsilon})(-1)}{(M^2-m^2)^2} \\
&= \frac{1}{(1-\epsilon)(2-\epsilon)}\frac{-(2-\epsilon)(m^2)^{1-\epsilon}(M^2-m^2)-((M^2)^{2-\epsilon}-(m^2)^{2-\epsilon})(-1)}{(M^2-m^2)^2} \\
&= \frac{1}{(1-\epsilon)(2-\epsilon)}\frac{-(2-\epsilon)(m^2)^{1-\epsilon}M^2+(1-\epsilon)(m^2)^{2-\epsilon}+(M^2)^{2-\epsilon}}{(M^2-m^2)^2}
\end{aligned}
\tag{39}
$$

Therefore,

$$
\begin{aligned}
\delta_{Z_\psi} &= -g^2\frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}}\frac{-(2-\epsilon)(m^2)^{1-\epsilon}M^2+(1-\epsilon)(m^2)^{2-\epsilon}+(M^2)^{2-\epsilon}}{(1-\epsilon)(2-\epsilon)(M^2-m^2)^2} \\
&= -g^2\frac{1}{(4\pi)^2}\frac{1}{\epsilon}\frac{1}{2}+\text{finite}.
\end{aligned}
\tag{40}
$$

Just like the scalar two-point function, we can explicitly verify that the two-point function is made manifestly finite with the counter terms. We subtracted the zeroth order and first order pieces in $\not p$ from Eq. (32), and hence

$$
\begin{aligned}
i\Gamma(\not p) &= i(\not p - M) - i\Sigma(\not p) + i\delta_{Z_\psi}\not p - i\delta_M \\
&= i(\not p - M) - g^2\frac{i\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}}\int_0^1 dz[-(1-z)\not p + M] \\
&\quad \left\{(1-z)m^2+zM^2-z(1-z)p^2\right]^{-\epsilon} - [(1-z)m^2+zM^2]^{-\epsilon}\} \\
&= i(\not p - M) + g^2\frac{i}{(4\pi)^2}\int_0^1 dz[-(1-z)\not p + M]\ln\frac{(1-z)m^2+zM^2-z(1-z)p^2}{(1-z)m^2+zM^2}
\end{aligned}
\tag{41}
$$

This expression is manifestly finite, while the location of the pole is not at $\not p = m$ nor is the residue unity. The same caution about S-matrix elements applies as in the scalar case.

2.3 Fermion-Fermion-Scalar Three-Point Function

The one-loop diagram for the vertex correction is

$$
-iV(p_f,p_i) = g^3\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2}\gamma^5\frac{i}{\not p_f+\not k - M}\gamma^5\frac{i}{\not p_i+\not k-M}\gamma^5. \tag{42}
$$

It is logarithmically divergent, which needs be cancelled by the counter term. We require that the full amputated three-point function

$$-i\Gamma(p_f, p_i) = g\gamma^5 - iV(p_f, p_i) + \delta_g\gamma^5 \tag{43}$$

is given by $g\gamma^5$ at zero momentum,

$$-i\Gamma(0,0) = g\gamma^5 - iV(0,0) + \delta_g\gamma^5 = g\gamma^5, \tag{44}$$

namely $\delta_g\gamma^5 - iV(0,0) = 0$.

We can now compute the counter term.

$$\begin{aligned}
\delta_g\gamma^5 &= iV(0,0) \\
&= -g^3 \int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2 - m^2}\gamma^5 \frac{i}{\not{k} - M}\gamma^5 \frac{i}{\not{k} - M}\gamma^5 \\
&= ig^3 \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 - m^2} \frac{\gamma^5(\not{k} + M)\gamma^5(\not{k} + M)\gamma^5}{[k^2 - M^2]^2} \\
&= ig^3 \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 - m^2} \frac{\gamma^5(-k^2 + M^2)}{[k^2 - M^2]^2} \\
&= -ig^3\gamma^5 \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 - m^2)(k^2 - M^2)} \\
&= -ig^3\gamma^5 \int \frac{d^dk}{(2\pi)^d} \int_0^1 dz \frac{1}{[k^2 - zm^2 - (1-z)M^2]^2} \\
&= -ig^3\gamma^5 \int_0^1 dz \frac{i\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} [zm^2 + (1-z)M^2]^{-\epsilon} \\
&= -ig^3\gamma^5 \frac{i\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} \left[\frac{1}{1-\epsilon} \frac{1}{m^2 - M^2} [z(m^2 - M^2) + M^2]^{1-\epsilon}\right]_0^1 \\
&= g^3\gamma^5 \frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} \frac{1}{1-\epsilon} \frac{(m^2)^{1-\epsilon} - (M^2)^{1-\epsilon}}{m^2 - M^2}.
\end{aligned} \tag{45}$$

Therefore,

$$\delta_g = g^3 \frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} \frac{1}{1-\epsilon} \frac{(m^2)^{1-\epsilon} - (M^2)^{1-\epsilon}}{m^2 - M^2} = g^3 \frac{1}{(4\pi)^2} \frac{1}{\epsilon} + \text{finite}. \tag{46}$$

2.4 Scalar Four-Point Function

The one-loop box diagram for the scalar four-point function is quite complicated. With four scalars with four-momenta p_1, p_2, p_3, and p_4, all defined to be coming into the diagram and subject to the overall four-momentum conservation $p_1^\mu + p_2^\mu + p_3^\mu + p_4^\mu = 0$, there are 4! ways of ordering them. However, the cyclic permutation does not give rise to new diagrams and hence 4!/4=3!=6 diagrams are independent. One can verify this also from the first principle by using Wick's theorem to contract scalars and fermions. All six digrams are lograrithmically divergent.

One of the diagrams is

$$-g^4 \int \frac{d^dk}{(2\pi)^d} \mathrm{Tr}\gamma^5 \frac{i}{\not{k} - M} \gamma^5 \frac{i}{\not{k} + \not{p}_1 - M} \gamma^5 \frac{i}{\not{k} + \not{p}_1 + \not{p}_2 - M} \gamma^5 \frac{i}{\not{k} - \not{p}_4 - M}. \quad (47)$$

The other diagrams are obtained by permuting the four-momenta.

However, if all external momenta vanish, all six diagrams become the same.

$$\begin{aligned} &-g^4 \int \frac{d^dk}{(2\pi)^d} \mathrm{Tr}\gamma^5 \frac{i}{\not{k} - M} \gamma^5 \frac{i}{\not{k} - M} \gamma^5 \frac{i}{\not{k} - M} \gamma^5 \frac{i}{\not{k} - M} \times 6 \\ &\quad = -6g^4 \int \frac{d^dk}{(2\pi)^d} \frac{\mathrm{Tr}\gamma^5(\not{k} + M)\gamma^5(\not{k} + M)\gamma^5(\not{k} + M)\gamma^5(\not{k} + M)}{[k^2 - M^2]^4}. \end{aligned} \quad (48)$$

The numerator simplifies to

$$\begin{aligned} &\mathrm{Tr}\gamma^5(\not{k} + M)\gamma^5(\not{k} + M)\gamma^5(\not{k} + M)\gamma^5(\not{k} + M) \\ &\quad = \mathrm{Tr}\gamma^5\gamma^5(-\not{k} + M)(\not{k} + M)\gamma^5\gamma^5(-\not{k} + M)(\not{k} + M) \\ &\quad = 4[k^2 - M^2]^2. \end{aligned} \quad (49)$$

Therefore, the sum of six diagrams with zero external momenta is

$$= -24g^4 \int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2 - M^2]^2} = -24g^4 \frac{i\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} (M^2)^{-\epsilon}. \quad (50)$$

To cancel this by the counter term $-i\delta_\lambda$, we need

$$\delta_\lambda = -24g^4 \frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} (M^2)^{-\epsilon} = -24g^4 \frac{1}{(4\pi)^2} \frac{1}{\epsilon} + \text{finite}. \quad (51)$$