HW #7 Physics 230A, Spring 2007, Hitoshi Murayama

16.1 Arnowitt–Fickler gauge (axial gauge).

To keep the notation managable, I use the gauge condition $n \cdot A^a = n^\mu A^a_\mu = 0$, where $n^\mu = (0,0,0,1)$ ($n^2 = -1$) is the choice in the problem. Using the by-now-familiar Faddeev–Popov method, we insert unity into the path integral,

$$1 = \int \mathcal{D}\alpha \delta(n \cdot A^a) \left| \det \frac{\delta(n \cdot A^a)}{\delta \alpha} \right| . \tag{1}$$

The infinitesimal gauge transformation is

$$\delta(n \cdot A^a) = n^\mu (D_\mu \alpha)^a = n^\mu (\partial_\mu \alpha - ig[A_\mu, \alpha])^a. \tag{2}$$

Note, however, that the Faddeev–Popov determinant multiplies the delta function that forces $n^\mu A_\mu = 0$, and hence the infinitesimal gauge transformation is

$$\delta(n \cdot A^\alpha) = n^\mu \partial_\mu \alpha^a. \tag{3}$$

Therefore, the unity we insert into the path integral is

$$1 = \int \mathcal{D}\alpha \delta(n \cdot A^a) \left| \det n^\mu \partial_\mu \right| . \tag{4}$$

It is clear that the determinant does not depend on the gauge field and gives only an overall normalization factor.

If we rewrite the determinant in terms of Faddeev–Popov ghosts, it is simply

$$\mathcal{L}_{FP} = \bar{c} n^\mu \partial_\mu c. \tag{5}$$

It does not propagate in the sense that its equation of motion is not a wave equation, simply $n^\mu \partial_\mu c = \partial_z c = 0$. Any function independent of z is a solution, and there is no sense that the past determines future. In any case, the ghosts do not interact with the gauge field and can simply be dropped.

The Feynman rule for the interaction of the gauge fields is the same as in the other gauges, *i.e.*, Figure 16.1 in the book. What is different is the propagator.

The kinetic term of the Lagrangian in the momentum space is

$$ik^2 \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) . \tag{6}$$

The reason we have to fix the gauge for the perturbation theory is that this matrix in the parentheses is not invertible. In the axial gauge, we do not need to invert this matrix on the full four-dimensional space, but only on the remaining three.

Let us specialize to the case at hand, $n^\mu = (0,0,0,1)$. Then we invert the matrix only on (t,x,y). We use the subscript $_\perp$ when the quantity is defined on this space orthogonal to n^μ. On the other hand, for $k^\mu = (k_0, k_x, k_y, k_z)$, $k^2 = k_0^2 - k_x^2 - k_y^2 - k_z^2$ which includes k_z. This is what allows us to invert the matrix. Explicitly,

$$\begin{aligned} ik^2 \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right)_\perp &= i \begin{pmatrix} -k_x^2 - k_y^2 - k_z^2 & k_0 k_x & k_0 k_y \\ k_0 k_x & -k_0^2 + k_y^2 + k_z^2 & k_x k_y \\ k_x k_y & -k_0^2 + k_x^2 + k_y^2 & k_y k_z \end{pmatrix} \\ &= ik_\perp^2 \left(g_\perp^{\mu\nu} - \frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2} \right) - ik_z^2 g_\perp^{\mu\nu} \\ &= ik^2 \left(g_\perp^{\mu\nu} - \frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2} \right) - ik_z^2 \frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2}. \end{aligned} \tag{7}$$

Here, $k_\perp^\mu = (k_0, k_x, k_y)$, $k_\perp^2 = k_0^2 - k_x^2 - k_y^2$ and $g_\perp^{\mu\nu}$ is the projection of the metric tensor on the orthogonal space of z.

The inverse is

$$\begin{aligned} \frac{-i}{k^2} \left(g_\perp^{\mu\nu} - \frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2} \right) + \frac{i}{k_z^2} \frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2} &= \frac{-i g_\perp^{\mu\nu}}{k^2} + \frac{i(k^2 + k_z^2)}{k^2 k_z^2} \frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2} \\ &= \frac{-i g_\perp^{\mu\nu}}{k^2} + \frac{i k_\perp^\mu k_\perp^\nu}{k^2 k_z^2} \\ &= \frac{i}{k^2} \left(-g_\perp^{\mu\nu} + \frac{k_\perp^\mu k_\perp^\nu}{k_z^2} \right). \end{aligned} \tag{8}$$

This is the gauge boson propagator in this gauge. It is easy to verify

$$\begin{aligned} \left(g_\perp^{\mu\nu} - \frac{k_\perp^\mu k_\perp^\nu}{k^2} \right) \left(g_{\perp\nu\rho} - \frac{k_{\perp\nu} k_{\perp\rho}}{k_z^2} \right) &= \delta_\rho^\mu - \frac{k_\perp^\mu k_{\perp\rho}}{k^2} - \frac{k_\perp^\mu k_{\perp\rho}}{k_z^2} + \frac{k_\perp^\mu k_{\perp\rho} k_\perp^2}{k^2 k_z^2} \\ &= \delta_\rho^\mu - \frac{k_\perp^\mu k_{\perp\rho}}{k^2 k_z^2} (k_z^2 + k^2 - k_\perp^2) = \delta_\rho^\mu. \end{aligned} \tag{9}$$

For general n^μ, the propagator is

$$P_\perp^{\mu\rho} \frac{i}{k^2} \left(-g_{\rho\sigma} + \frac{k_\rho k_\sigma}{(n \cdot k)^2} \right) P_\perp^{\sigma\nu}, \qquad P_\perp^{\mu\rho} = g^{\mu\rho} - \frac{n^\mu n^\rho}{n^2}. \tag{10}$$

It can be further written out by summing over ρ, σ indices and we find

$$\frac{-i}{k^2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{(k\cdot n)^2} - \frac{k^\mu n^\nu + k^\nu n^\mu}{k\cdot n}\right). \tag{11}$$

Another way to obtain the propagator is to take the limit $\lambda \to \infty$ starting with the Lagrangian

$$\mathcal{L}_2 = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \frac{\lambda}{2}(n\cdot A)^a(n\cdot A)^a. \tag{12}$$

For a finite λ, the propagator is

$$\frac{-i}{k^2}\left(g^{\mu\nu} + \frac{(k^2+\lambda n^2)k^\mu k^\nu}{\lambda(k\cdot n)^2} - \frac{k^\mu n^\nu + k^\nu n^\mu}{k\cdot n}\right). \tag{13}$$

It has a bad high-energy behavior $O(k^0)$. However with the limit $\lambda \to \infty$, we find a propagator that is well-behaved,

$$\frac{-i}{k^2}\left(g^{\mu\nu} + \frac{n^2 k^\mu k^\nu}{(k\cdot n)^2} - \frac{k^\mu n^\nu + k^\nu n^\mu}{k\cdot n}\right). \tag{14}$$

This is exactly the same as what we had worked out above using $n^2 = -1$.

The propagator appears singular when $k_z \to 0$. However, this does not lead to a singularity thanks to the gauge invariance. If we use this propagator for a simple process such as the virtual photon in $e^+e^- \to \mu^+\mu^-$, the current conservation for both electron and muon lets us replace $k_\perp^\mu \to k_\perp^\mu - k^\mu = (0,0,0,-k_z)$. Therefore the apparent singularity is now resolved to $(-k_z)(-k_z)/k_z^2 = 1$. Even for more complicated processes, this replacement is possible after all diagrams need to be added together and the external lines taken on-shell, and the singularity disappears in the end.

The gauge boson states now come in three polarization vectors, one of which is removed by the residual gauge transformation. The axial gauge allows us to make a gauge transformation by any z-independent function. Therefore it still allows us to change the other components proportional to $k_\perp^\mu$. This reduces the number of physical polarization vectors to two. Explicitly for $k^\mu = k_0(1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, the three independent polarization vectors are

$$\epsilon^\mu_{\perp,L} = \frac{1}{\cos\theta}(1, \sin\theta\cos\phi, \sin\theta\sin\phi) \tag{15}$$

$$\epsilon^\mu_{\perp,1} = (0, \sin\phi, -\cos\phi) \tag{16}$$

$$\epsilon^\mu_{\perp,2} = \frac{1}{\cos\theta}(\sin\theta, \cos\phi, \sin\phi). \tag{17}$$

The first one gives a negative norm state but is removed by the residual gauge transformation, while the latter two give the physical polarization vectors.

It is very important to know that there is a gauge that does not need any Faddeev–Popov ghosts. In the usual Lorentz-invariant gauges, we have the condition that "physical states" satisfy

$$Q_{BRST}|\text{phys}\rangle = 0. \tag{18}$$

However, any state that can be written as

$$|\text{phys}\rangle = Q_{BRST}|\text{unphysical}\rangle \tag{19}$$

satisfies the physical state condition. In other words, there are infinitely many "physical" states with ghosts and anti-ghosts, which sound worriesome. However, these states are clearly not truly physical, as their norms vanish,

$$|Q_{BRST}|\text{unphysical}\rangle|^2 = \langle\text{unphysical}|Q_{BRST}Q_{BRST}|\text{unphysical}\rangle = 0. \tag{20}$$

The last steps follows from the nilpotency of the BRST charge $Q_{BRST}^2 = 0$. The only states that are truly physical are those annihilated by Q_{BRST} but cannot be written as Q_{BRST} acting on another state. The existence of a gauge that does not require any ghosts implies that the truly physical states do not contain any ghosts or anti-ghosts.[1]

16.2 Scalar field with non-Abelian charge.

(a)

The interaction of a complex scalar field to gauge bosons comes from the kinetic term the Lagrangian

$$(D_\mu\phi)^\dagger(D^\mu\phi) = (\partial_\mu\phi^\dagger + ig\phi^\dagger A_\mu)(\partial^\mu\phi - igA^\mu\phi), \tag{21}$$

where $A_\mu = A_\mu^a t^a$ is written as a matrix and ϕ a column vector. Expanding it out, we find

$$\partial_\mu\phi^\dagger\partial^\mu\phi - ig(\partial_\mu\phi^\dagger A^\mu\phi - \phi^\dagger A^\mu\partial_\mu\phi) + g^2\phi^\dagger A_\mu A^\mu\phi. \tag{22}$$

[1] Namely, the BRST coholomology is only in the sector of the Hilbert space with zero ghost number.

From this, we can read off the Feynman rules.

Relative to the Feynman rules in Problem 9.1(a), the first vertex (scalar-scalar-vector) comes with the generator $-igt^a(p+p')^\mu$. The second vertex (scalar-scalar-vector-vector) comes from the last term, and is therefore $ig^2\{t^a, t^b\}g^{\mu\nu}$ instead of $2ie^2g^{\mu\nu}$.

(b)

This is mostly the same calculation as the part (e).1 in HW #5, which gave us

$$\beta(e) = \frac{e^3}{48\pi^2}. \tag{23}$$

The difference is the group theory factor $\mathrm{tr} t^a t^b = C(r)\delta^{ab}$. Therefore, we find the scalar field contribution to the beta function to be

$$+\frac{g^3}{48\pi^2}C(r). \tag{24}$$

Adding it to the contribution of the gauge bosons (and associated Faddeev–Popov ghosts), we find the result as desired,

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}C_2(G) - \frac{1}{3}C(r)\right). \tag{25}$$

Note that the complex scalar field contributes 1/4 of that of Dirac fermion.