HW #6 Physics 230A, Spring 2007, Hitoshi Murayama

# 15.3 Coulomb potential.

## (a)

This is a straightforward generalization of Problem 11.1 we studied in HW #4. We include the Wilson line in to the action in the path integral using the source, and the result is given by the two powers of the source with the propagator in between. Therefore, what we need is the propagator which we compute in the Feynman gauge. In the coordinate space,

$$\begin{aligned} D^{\mu\nu}(x) &= \langle 0|TA^\mu(x)A^\nu(0)|0\rangle \\ &= \int \frac{d^4q}{(2\pi)^4}\frac{-ig^{\mu\nu}e^{-iq\cdot x}}{q^2+i\epsilon} \\ &= \int_0^\infty d\tau \int \frac{d^4q}{(2\pi)^4}(-g^{\mu\nu})e^{-iq\cdot x}e^{i\tau(q^2+i\epsilon)} \\ &= -g^{\mu\nu}\int_0^\infty d\tau \int \frac{d^4q}{(2\pi)^4} e^{i\tau(q-x/2\tau)^2}e^{-ix^2/4\tau}e^{-\epsilon\tau} \\ &= -g^{\mu\nu}\int_0^\infty d\tau \frac{1}{(2\pi)^4}\left(\frac{\pi}{-i\tau}\right)^{1/2}\left(\frac{\pi}{+i\tau}\right)^{3/2} e^{-ix^2/4\tau}e^{-\epsilon\tau} \\ &= ig^{\mu\nu}\int_0^\infty d\tau \frac{1}{(4\pi)^2}\frac{1}{\tau^2}e^{-ix^2/4\tau}e^{-\epsilon\tau} \\ &= ig^{\mu\nu}\int_0^\infty ds \frac{1}{(4\pi)^2}e^{-isx^2/4}e^{-\epsilon/s}\ . \end{aligned} \tag{1}$$

Because the $\epsilon$ prescription is there just to specify how the pole is avoided, we only need to retain the correct sign for $\epsilon$ and hence we can replace $\epsilon \to \epsilon s^2/4$. Then we find

$$\begin{aligned} D^{\mu\nu}(x) &= ig^{\mu\nu}\int_0^\infty ds \frac{1}{(4\pi)^2}e^{-is(x^2-i\epsilon)/4} \\ &= ig^{\mu\nu}\frac{1}{(4\pi)^2}\frac{-4i}{x^2-i\epsilon} \\ &= \frac{1}{4\pi^2}\frac{g^{\mu\nu}}{x^2-i\epsilon} \end{aligned} \tag{2}$$

The path integral with the Wilson line then reduces to

$$\begin{aligned}\langle e^{-ie\oint_P A_\mu dx^\mu}\rangle &= \exp\left[\frac{1}{2}(-ie)^2\oint_P dx^\mu \oint_P dy^\nu \langle 0|TA_\mu(x)A_\nu(y)|0\rangle\right] \\ &= \exp\left[-\frac{1}{2}e^2\oint_P dx^\mu \oint_P dy^\nu \frac{1}{4\pi^2}\frac{g_{\mu\nu}}{(x-y)^2-i\epsilon}\right] \\ &= \exp\left[-e^2\oint_P dx^\mu \oint_P dy^\nu \frac{g_{\mu\nu}}{8\pi^2[(x-y)^2-i\epsilon]}\right]. \qquad (3)\end{aligned}$$

This is what is desired, except for the fact that we've carefully worked out how the pole is avoided in this integral.

## (b)

The loop $P$ consists of four portions, $C_1$, $C_2$, $C_3$, $C_4$ as show in Fig. 1.

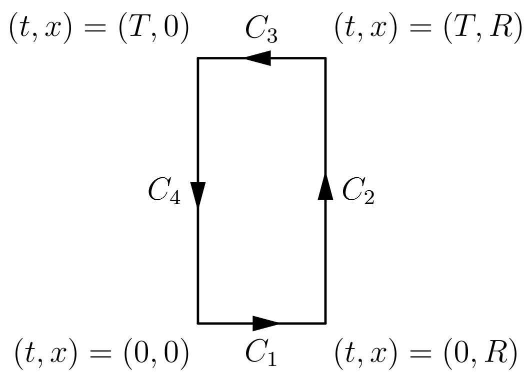


Figure 1: The loop along which the Wilson loop is defined.

In the limit $T \gg R$, clearly the most important contributions come from the paths $C_2$ and $C_4$. We also avoid the self-energy contributions, namely those from the same paths. We will take the $T \to \infty$ limit while keeping $R$ finite. The exponent of the Wilson loop is then

$$\begin{aligned}&-2e^2\int_{C_2} dx^\mu \int_{C_4} dy^\nu \frac{g_{\mu\nu}}{8\pi^2[(x-y)^2-i\epsilon]} \\ &= -2e^2\int_0^T dx^0 \int_T^0 dy^0 \frac{g_{00}}{8\pi^2[(x^0-y^0)^2-R^2-i\epsilon]}\end{aligned}$$

$$
\begin{aligned}
&\approx -\frac{e^2}{4\pi^2} T \int_{\infty}^{-\infty} dy^0 \frac{1}{(y^0)^2 - R^2 - i\epsilon} \\
&= +\frac{e^2}{4\pi^2} T \int_{-\infty}^{\infty} dy^0 \frac{1}{(y^0 - R - i\epsilon)(y^0 + R + i\epsilon)} \\
&= \frac{e^2}{4\pi^2} T \, 2\pi i \frac{1}{2R} \\
&= i \frac{e^2}{4\pi R} T \ .
\end{aligned} \tag{4}
$$

For the contour integral used in the second last step, see, *e.g.*, Lecture note on Contour Integrals. This is the desired result $-iV(R)T$ with the Coulomb potential $V(R) = -\frac{e^2}{4\pi R}$.

### (c)

The only difference from the abelian case above is that the gauge field in the Wilson loop is $A_\mu = A_\mu^a t_r^a$, where $t_r^a$ is the generator in the representation $r$. Because the propagator at the lowest order in $g$ is $\langle 0|T A_\mu^a(x) A_\nu^b(0)|0\rangle = D_{\mu\nu}(x)\delta^{ab}$, the exponent is proportional to $\delta^{ab} t_r^a t_r^b = C_2(r)$. Therefore, we simply replace $e^2$ by $g^2 C_2(r)$, and hence the potential is $V(R) = -\frac{g^2 C_2(r)}{4\pi R}$.

## 15.5 Casimir operator computations

### (a)

If you are not familiar with the tensor products and direct sum of vector spaces, consult the 221A lecture note *Tensor product and direct sum.*

If the irreducible representation $r$ decomposes as

$$
r \to \sum_{i=1}^{k} j_i = j_1 \oplus j_2 \oplus \cdots \oplus j_k \tag{5}
$$

under $SU(2) \subset G$, the generators of the $SU(2)$ are given by

$$
t_r^a = t_{j_1}^a \oplus t_{j_2}^a \oplus \cdots \oplus t_{j_k}^a. \tag{6}
$$

Here, $a = 1, 2, 3$ for the $SU(2)$ generators, $t_r^a$ are the subset of generators of $G$ in the representation $r$, and $t_{j_i}^a$ the $SU(2)$ generators in the spin $j_i$

representation. Given this decomposition,

$$\mathrm{tr}(t_r^a t_r^b) = \mathrm{tr}(t_{j_1}^a t_{j_1}^b) + \mathrm{tr}(t_{j_2}^a t_{j_2}^b) + \cdots \mathrm{tr}(t_{j_k}^a t_{j_k}^b). \tag{7}$$

For each traces in spin $j$ representation, we know that the Casimir operator is $C_2(j) \sum_{a=1}^{3} t_j^a t_j^a = j(j+1)$, while the representation is $d(j) = 2j+1$ dimensional. This allows us to compute

$$\mathrm{tr}(t_j^a t_j^b) = C(j)\delta^{ab}. \tag{8}$$

By summing over $a = b$, the matrix inside the trace is nothing but the Casimir operator,

$$\sum_{a=1}^{3} \mathrm{tr}(t_j^a t_j^b) = \mathrm{tr} j(j+1) = j(j+1)(2j+1) \tag{9}$$

Note that in the second step $j(j+1)$ multiplies the unit matrix in the $2j+1$ dimensional space. The r.h.s. gives

$$C(j) \sum_{a=1}^{3} \delta^{aa} = 3C(j). \tag{10}$$

Therefore, we find

$$3C(j) = j(j+1)(2j+1). \tag{11}$$

This technique is nothing but Eq. (15.94) in the book, using $d(SU(2)) = 3$. Going back to Eq. (7), we find

$$\mathrm{tr}(t_r^a t_r^b) = \sum_{i=1}^{k} \frac{1}{3} j_i(j_i+1)(2j_i+1)\delta^{ab}, \tag{12}$$

and hence we obtain

$$3C(r) = \sum_i j_i(j_i+1)(2j_i+1) \tag{13}$$

as desired.

## (b)

The $N$-dimensional representation of $SU(N)$ decomposes under $SU(2) \subset SU(N)$ as

$$N \to 2 \oplus \underbrace{1 \oplus \cdots \oplus 1}_{N-2}. \tag{14}$$

Here, the notation "2" refers to 2-dimensional representation of $SU(2)$, namely $j = 1/2$. It is confusing, but people switch back and force between $j$ and $2j+1$. Using the result from the previous part, we find

$$3C(N) = \frac{1}{2}(\frac{1}{2}+1)(2\frac{1}{2}+1) + \underbrace{0 + \cdots + 0}_{N-2} = \frac{3}{2}. \tag{15}$$

Therefore,

$$C(N) = \frac{1}{2}. \tag{16}$$

This is the standard normalization of generators among physicists, $\mathrm{tr}(t^a_N t^b_N) = \frac{1}{2}\delta^{ab}$.

The adjoint representation is found in the tensor product of $N$ and its conjugate representation $\overline{N}$ for $SU(N)$ groups: $N \otimes \overline{N} = \text{adjoint} \oplus 1$. You can understand it this way. The $N$ representation is a column vector $\mathbf{v}$ that transforms as $\mathbf{v} \to U\mathbf{v}$. The $\overline{N}$ representation is another column vector $\mathbf{w} \to U^*\mathbf{w}$. The singlet (and hence invariant) is nothing but $\mathbf{w}^T\mathbf{v} \to \mathbf{w}^T U^\dagger U \mathbf{v} = \mathbf{w}^T\mathbf{v}$. The ajoint representation is nothing but $\mathbf{w}^T t^a \mathbf{v}$, which transforms to $\mathbf{w}^T U^\dagger t^a U \mathbf{v} = R^{ab}\mathbf{w}^T t^b \mathbf{v}$, where $R^{ab} t^b = U^\dagger t^a U$ is how the generators transform, which is by definition what the adjoint representation is.

Under the decomposition to $SU(2)$,

$$\begin{aligned} N \otimes \overline{N} &= (2 \oplus \underbrace{1 \oplus \cdots \oplus 1}_{N-2}) \otimes (2 \oplus \underbrace{1 \oplus \cdots \oplus 1}_{N-2}) \\ &= (2 \otimes 2) \oplus (2 \otimes (\underbrace{1 \oplus \cdots \oplus 1}_{N-2})) \oplus ((\underbrace{1 \oplus \cdots \oplus 1}_{N-2}) \otimes 2) \\ &\quad \oplus ((\underbrace{1 \oplus \cdots \oplus 1}_{N-2}) \otimes (\underbrace{1 \oplus \cdots \oplus 1}_{N-2})) \\ &= (3 \oplus 1) \oplus \underbrace{2 \oplus \cdots \oplus 2}_{2(N-2)} \oplus \underbrace{(1 \oplus \cdots \oplus 1)}_{(N-2)^2}. \end{aligned} \tag{17}$$

A quick sanity check: the dimension of space originally is $N^2$, and after the decomposition, it is $3 + 1 + 2(N-2) \times 2 + (N-2)^2 = N^2$. Good. Then using the result of part (a),

$$3C(\text{adj}) = (2\cdot1+1)1(1+1)+0+(2\cdot\frac{1}{2}+1)\frac{1}{2}(\frac{1}{2}+1)(N-2)2+((N-2)^2-1)0 = 3N. \tag{18}$$

Therefore, we find $C(\text{adj}) = N$ for $SU(N)$.

## (c)

Now we look at the symmetric and anti-symmetric two-index tensor representations. They appear in the tensor product of two fundamental representations $N \otimes N = S \oplus A$, where $S = (N \otimes N)_{sym}$ is a symmetric combination of two $N$'s and hence $N(N+1)/2$ dimensional, while $A = (N \otimes N)_{asym}$ is an anti-symmetric combination which is $N(N-1)/2$ dimensional. Using the same procedure as in part (b), we look at the symmetric combination first,

$$\begin{aligned}(N \otimes N)_{sym} &= [(2 \oplus \underbrace{1 \oplus \cdots \oplus 1}_{N-2}) \otimes (2 \oplus \underbrace{1 \oplus \cdots \oplus 1}_{N-2})]_{sym} \\ &= (2 \otimes 2)_{sym} \oplus [2 \otimes (\underbrace{1 \oplus \cdots \oplus 1}_{N-2})]_{sym} \oplus [\underbrace{1 \oplus \cdots \oplus 1}_{N-2} \otimes \underbrace{1 \oplus \cdots \oplus 1}_{N-2}]_{sym} \\ &= 3 \oplus (\underbrace{2 \oplus \cdots \oplus 2}_{N-2}) \oplus \underbrace{1 \oplus \cdots \oplus 1}_{(N-2)(N-1)/2} .\end{aligned} \tag{19}$$

Again using the result from part (a), we find

$$3C(S) = (2 \cdot 1 + 1)1(1+1) + (2 \cdot \frac{1}{2} + 1)\frac{1}{2}(\frac{1}{2} + 1)(N-2) = 3\frac{N+2}{2}, \tag{20}$$

and hence $C(S) = \frac{N+2}{2}$. Then using Eq. (15.94) in the book, we find

$$C_2(S) = C(S)\frac{d(G)}{d(S)} = \frac{N+2}{2}\frac{N^2-1}{N(N+1)/2} = \frac{(N+2)(N-1)}{N} . \tag{21}$$

For the anti-symmetric two-index tensor,

$$\begin{aligned}(N \otimes N)_{asym} &= [(2 \oplus \underbrace{1 \oplus \cdots \oplus 1}_{N-2}) \otimes (2 \oplus \underbrace{1 \oplus \cdots \oplus 1}_{N-2})]_{asym} \\ &= (2 \otimes 2)_{asym} \oplus [2 \otimes (\underbrace{1 \oplus \cdots \oplus 1}_{N-2})]_{asym} \oplus [\underbrace{1 \oplus \cdots \oplus 1}_{N-2} \otimes \underbrace{1 \oplus \cdots \oplus 1}_{N-2}]_{asym} \\ &= 1 \oplus (\underbrace{2 \oplus \cdots \oplus 2}_{N-2}) \oplus \underbrace{1 \oplus \cdots \oplus 1}_{(N-2)(N-3)/2} .\end{aligned} \tag{22}$$

Again using the result from part (a), we find

$$3C(A) = (2 \cdot \frac{1}{2} + 1)\frac{1}{2}(\frac{1}{2} + 1)(N - 2) = 3\frac{N-2}{2}, \tag{23}$$

and hence $C(A) = \frac{N-2}{2}$. Then using Eq. (15.94) in the book, we find

$$C_2(A) = C(A)\frac{d(G)}{d(S)} = \frac{N-2}{2}\frac{N^2-1}{N(N-1)/2} = \frac{(N-2)(N+1)}{N} \ . \tag{24}$$

Eq. (15.100) tells us that

$$\mathrm{tr}(t^a_{N\otimes N})^2 = (C_2(N)+C_2(N))d(N)d(N) = 2\frac{N^2-1}{2N}NN = N(N+1)(N-1). \tag{25}$$

On the other hand, Eq. (15.101) tell us that

$$\begin{aligned}\mathrm{tr}(t^a_{N\otimes N})^2 &= C_2(S)d(S) + C_2(A)d(A) \\ &= \frac{(N+2)(N-1)}{N}\frac{N(N+1)}{2} + \frac{(N-2)(N+1)}{N}\frac{N(N-1)}{2} \\ &= N(N+1)(N-1).\end{aligned} \tag{26}$$

Therefore, our results are consistent with the general formulae Eqs. (15.100) and (15.101).