Spin-Wave theory and Asymptotically Free Beta-Function

Physics 230A, Spring 2007, Hitoshi Murayama

Problem 11.1

(a)

The free scalar field theory is defined by the path integral

$$Z = \int \mathcal{D}\phi(x) e^{i\int d^D x[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2]}. \tag{1}$$

Any vacuum expectation value of a time-ordered product of operators is given in terms of the path integral

$$\langle 0|T\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)|0\rangle = \frac{1}{Z}\int \mathcal{D}\phi(x) e^{i\int d^D x[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2]} O_1(x_1)\cdots O_n(x_n). \tag{2}$$

The correlation function in the problem is therefore

$$\langle 0|Te^{i\phi(y)}e^{-i\phi(z)}|0\rangle = \frac{1}{Z}\int \mathcal{D}\phi(x) e^{i\int d^D x[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2]} e^{i\phi(y)}e^{-i\phi(z)}. \tag{3}$$

(I've changed the arguments to y and z for later convenience.) Note that it can be rewritten in terms of a source $J(z)$

$$\langle 0|Te^{i\phi(y)}e^{-i\phi(z)}|0\rangle = \frac{1}{Z}\int \mathcal{D}\phi(x) e^{i\int d^D x\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + J(x)\phi(x)\right)}, \tag{4}$$

where

$$J(x) = \delta^D(x-y) - \delta^D(x-z). \tag{5}$$

This is the path integral worked out in Section 9.2 of Peskin–Schroeder. Namely, we rewrite the action as ("complete the square")

$$\begin{aligned} S &= \int d^D x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + J\phi\right] \\ &= \int d^D x \left[-\frac{1}{2}\phi(\partial_\mu\partial^\mu + m^2)\phi + J\phi\right] \end{aligned}$$

$$= \int d^D x \left[-\frac{1}{2} \left(\phi - (\partial^2 + m^2)^{-1} J\right) (\partial^2 + m^2) \left(\phi - (\partial^2 + m^2)^{-1} J\right) \right.$$
$$\left. + \frac{1}{2} J(\partial^2 + m^2)^{-1} J \right]. \tag{6}$$

In the second line, we integrated the Lagrangian by parts and dropped the surface terms assuming all local fields damp at the infinity. Note that the formal expression $(\partial^2 + m^2)^{-1}$ means

$$(\partial^2 + m^2)^{-1} J(x) = \int d^D y i D_F(x - y) J(y), \tag{7}$$

with the Feynman propagator

$$D_F(x - y) = \int \frac{d^D k}{(2\pi)^D} \frac{i e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon} \ . \tag{8}$$

Shifting the integration variable

$$\phi \to \phi + (\partial^2 + m^2)^{-1} J, \tag{9}$$

the path integral becomes

$$\frac{1}{Z} \int \mathcal{D}\phi(x) e^{i \int d^D x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + \frac{1}{2} J (\partial^2 + m^2)^{-1} J \right)} = e^{i \int d^D x \frac{1}{2} J (\partial^2 + m^2)^{-1} J}. \tag{10}$$

Now recalling Eq. (5), the exponent is

$$i \int d^D x \frac{1}{2} J(\partial^2 + m^2)^{-1} J = i \int d^D x d^D w \frac{1}{2} J(x) i D_F(x - w) J(w)$$
$$= -\frac{1}{2} (D_F(y - y) - D_F(y - z) - D_F(z - y) + D_F(z - z))$$
$$= D_F(y - z) - D_F(0). \tag{11}$$

Therefore, we find

$$\langle 0 | T e^{i\phi(y)} e^{-i\phi(z)} | 0 \rangle = e^{D_F(y-z) - D_F(0)} \tag{12}$$

as desired.

Other ways of showing this result are substantially more complicated. One way is to write the field operators in terms of creation and annihilation operators. Because this is a free field theory, we can write $\phi(x) = \phi_+(x) + \phi_-(x)$, with

$$\phi_+(x) = \int \frac{d^{D-1} \vec{p}}{(2\pi)^{D-1} 2E(p)} a(p) e^{-i(E(p)x^0 - \vec{p}\cdot\vec{x})}, \qquad \phi_-(x) = \phi_+(x)^\dagger. \tag{13}$$

Here, $E(p) = \sqrt{\vec{p}^2 + m^2}$. It is important to know their commutator,

$$\begin{aligned}
&[\phi_+(x), \phi_-(y)] \\
&\quad = \int \frac{d^{D-1}\vec{p}}{(2\pi)^{D-1} 2E(p)} \frac{d^{D-1}\vec{q}}{(2\pi)^{D-1} 2E(q)} [a(p) e^{-i(E(p)x^0 - \vec{p}\cdot\vec{x})}, a^\dagger(q) e^{i(E(q)y^0 - \vec{q}\cdot\vec{y})}] \\
&\quad = \int \frac{d^{D-1}\vec{p}}{(2\pi)^{D-1} 2E(p)} e^{-i(E(p)(x^0-y^0) - \vec{p}\cdot(\vec{x}-\vec{y}))} = D_+(x-y).
\end{aligned} \tag{14}$$

Note $(D_+(x-y))^* = D_+(y-x)$. More importantly, it is a number, not an operator. This point allows us to use the Campbell–Baker–Hausdorff formula

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]} \tag{15}$$

if $[X, [X, Y]] = [Y, [X, Y]] = 0$. Therefore, we find

$$e^{i\phi(x)} = e^{i\phi_-(x)} e^{i\phi_+(x)} e^{\frac{1}{2}[\phi_-(x), \phi_+(x)]} = e^{i\phi_-(x)} e^{i\phi_+(x)} e^{-\frac{1}{2}D_+(0)}. \tag{16}$$

Similarly,

$$e^{-i\phi(y)} = e^{-i\phi_-(y)} e^{-i\phi_+(y)} e^{\frac{1}{2}[\phi_-(y), \phi_+(y)]} = e^{-i\phi_-(y)} e^{-i\phi_+(y)} e^{-\frac{1}{2}D_+(0)}. \tag{17}$$

Then the problem is to work out

$$\langle 0|T e^{i\phi(x)} e^{-i\phi(y)}|0\rangle = \langle 0|T e^{i\phi_-(x)} e^{i\phi_+(x)} e^{-i\phi_-(y)} e^{-i\phi_+(y)}|0\rangle e^{-D_+(0)}. \tag{18}$$

Now we study two possible time orderings separately. For $x^0 > y^0$,

$$\begin{aligned}
&\langle 0|T e^{i\phi(x)} e^{-i\phi(y)}|0\rangle = \langle 0|e^{i\phi(x)} e^{-i\phi(y)}|0\rangle \\
&\quad = \langle 0|e^{i\phi_-(x)} e^{i\phi_+(x)} e^{-i\phi_-(y)} e^{-i\phi_+(y)}|0\rangle e^{-D_+(0)} = \langle 0|e^{i\phi_+(x)} e^{-i\phi_-(y)}|0\rangle e^{-D_+(0)}.
\end{aligned} \tag{19}$$

We used the fact that the annihilation operator annihilates the vacuum.

Recall for a harmonic oscillator, the state created by an exponential of the creation operator is a coherent state

$$|f\rangle = e^{fa^\dagger}|0\rangle = \sum_{n=0}^{\infty} \frac{f^n}{\sqrt{n!}}|n\rangle, \tag{20}$$

whose normalization is

$$\langle g|f\rangle = \langle 0|e^{g^* a} e^{fa^\dagger}|0\rangle = \sum_{m,n=0}^{\infty} \langle m|\frac{g^{*m}}{\sqrt{m!}} \frac{f^n}{\sqrt{n!}}|n\rangle = e^{g^* f}. \tag{21}$$

Our case at hand is a straight-forward generalization,

$$\begin{aligned}
&\langle 0|T e^{i\phi(x)} e^{-i\phi(y)}|0\rangle = \langle 0|e^{i\phi_+(x)} e^{-i\phi_-(y)}|0\rangle e^{-D_+(0)} \\
&\quad = \exp\left[\int \frac{d^{D-1}\vec{p}}{(2\pi)^{D-1} 2E(p)} (ie^{-i(E(p)x^0 - \vec{p}\cdot\vec{x})})(-ie^{i(E(p)y^0 - \vec{p}\cdot\vec{y})})\right] e^{-D_+(0)} \\
&\quad = e^{D_+(x-y) - D_+(0)}.
\end{aligned} \tag{22}$$

For the other time ordering $x^0 < y^0$, we find

$$\langle 0|Te^{i\phi(x)}e^{-i\phi(y)}|0\rangle = \langle 0|e^{-i\phi_+(x)}e^{i\phi_-(x)}|0\rangle e^{-D_+(0)} = e^{D_+(y-x)-D_+(0)}. \tag{23}$$

Putting the two cases together,

$$\langle 0|Te^{i\phi(x)}e^{-i\phi(y)}|0\rangle = \theta(x^0-y^0)e^{D_+(x-y)-D_+(0)} + \theta(y^0-x^0)e^{D_+(y-x)-D_+(0)}. \tag{24}$$

Now we have to relate $D_+(x-y)$ and $D_F(x-y)$. Using the definition of the Feynman propagator, we find

$$\begin{aligned}
D_F(x-y) &= \int \frac{d^D p}{(2\pi)^D}\frac{ie^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon} \\
&= \int \frac{dE}{2\pi}\frac{d^{D-1}\vec{p}}{(2\pi)^{D-1}}\frac{ie^{-iE(x^0-y^0)+i\vec{p}\cdot(\vec{x}-\vec{y})}}{(E-E(p)+i\epsilon)(E+E(p)-i\epsilon)} \\
&= \theta(x^0-y^0)\int \frac{d^{D-1}\vec{p}}{(2\pi)^{D-1}}\frac{e^{-iE(p)(x^0-y^0)+i\vec{p}\cdot(\vec{x}-\vec{y})}}{2E(p)} \\
&\quad -\theta(y^0-x^0)\int \frac{d^{D-1}\vec{p}}{(2\pi)^{D-1}}\frac{e^{-iE(p)(y^0-x^0)+i\vec{p}\cdot(\vec{x}-\vec{y})}}{-2E(p)} \\
&= \theta(x^0-y^0)\int \frac{d^{D-1}\vec{p}}{(2\pi)^{D-1}}\frac{e^{-iE(p)(x^0-y^0)+i\vec{p}\cdot(\vec{x}-\vec{y})}}{2E(p)} \\
&\quad +\theta(y^0-x^0)\int \frac{d^{D-1}\vec{p}}{(2\pi)^{D-1}}\frac{e^{-iE(p)(y^0-x^0)+i\vec{p}\cdot(\vec{y}-\vec{x})}}{2E(p)} \\
&= \theta(x^0-y^0)D_+(x-y) + \theta(y^0-x^0)D_+(y-x).
\end{aligned} \tag{25}$$

Therefore, Eq. (24) is simply

$$\langle 0|Te^{i\phi(x)}e^{-i\phi(y)}|0\rangle = e^{D_F(x-y)-D_F(0)}. \tag{26}$$

The other way is to expand the exponential in Taylor series and use Wick's theorem to compute each term. This is probably the first one that comes to mind, but it is very complicated. We first Taylor expand the exponentials,

$$\langle 0|Te^{i\phi(x)}e^{-i\phi(y)}|0\rangle = \sum_{n,m=0}^{\infty}\frac{i^n(-i)^m}{n!m!}\langle 0|T\phi(x)^n\phi(y)^m|0\rangle. \tag{27}$$

First of all, $n+m$ must be even to get a non-vanishing correlation function. Therefore, either both n and m are even, or both odd.

Let us look at the case $n = 2k \geq m = 2l$. We can contract $2j$ pairs between $\phi(x)$ and $\phi(y)$. There are ${}_{2k}C_{2j}\ {}_{2l}C_{2j}\ (2j)!$ choices for this. The remaining $2k-2j$ $\phi(x)$'s are contracted among each other, with $(2k-2j)!/(2!)^{k-j}/(k-j)!$ ways of dividing them up into pairs. Similarly there are $(2l-2j)!/(2!)^{l-j}/(l-j)!$ ways to divide up $2l-2j$ $\phi(y)$'s into $l-j$ pairs. Therefore,

$\langle 0|T\phi(x)^{2k}\phi(y)^{2l}|0\rangle$

$$= \sum_{j=0}^{l} {}_{2k}C_{2j}\ {}_{2l}C_{2j}(2j)! \frac{(2k-2j)!}{(2!)^{k-j}(k-j)!} \frac{(2l-2j)!}{(2!)^{l-j}(l-j)!} D_F(x-x)^{k-j} D_F(x-y)^{2j} D_F(y-y)^{l-j}$$

$$= \sum_{j=0}^{l} \frac{(2k)!(2l)!}{(2j)!2^{k+l-2j}(k-j)!(l-j)!} D_F(x-x)^{k-j} D_F(x-y)^{2j} D_F(y-y)^{l-j}. \tag{28}$$

When $n = 2k \leq m = 2l$, we just switch l and k, or change the sum to $\min(k,l)$. On the other hand, when $n = 2k+1 \geq m = 2l+1$, we contract $2j+1$ pairs between $\phi(x)$ and $\phi(y)$. There are ${}_{2k+1}C_{2j+1}$ choices for this, with $(2k-2j)!/(2!)^{k-j}/(k-j)!$ ways of dividing up the remaining $\phi(x)$ into pairs, and $(2l-2j)!/(2!)^{l-j}/(l-j)!$ for pairing $\phi(y)$. Therefore,

$$\begin{aligned}
&\langle 0|T\phi(x)^{2k+1}\phi(y)^{2l+1}|0\rangle \\
&\quad = \sum_{j=0}^{l} {}_{2k+1}C_{2j+1}\ {}_{2l+1}C_{2j+1}\ (2j+1)! \frac{(2k-2j)!}{(2!)^{k-j}(k-j)!} \frac{(2l-2j)!}{(2!)^{l-j}(l-j)!} \\
&\qquad\quad D_F(x-x)^{k-j} D_F(x-y)^{2j+1} D_F(y-y)^{l-j} \\
&\quad = \sum_{j=0}^{l} \frac{(2k+1)!(2l+1)!}{(2j+1)!2^{k+l-2j}(k-j)!(l-j)!} D_F(x-x)^{k-j} D_F(x-y)^{2j+1} D_F(y-y)^{l-j}.
\end{aligned} \tag{29}$$

Going back to the correlation function, we find

$$\begin{aligned}
&\langle 0|Te^{i\phi(x)}e^{-i\phi(y)}|0\rangle \\
&\quad = \sum_{k,l=0}^{\infty} \sum_{j=0}^{\min(k,l)} \frac{(-1)^{k+l}}{(2j)!2^{k+l-2j}(k-j)!(l-j)!} D_F(x-y)^{2j} D_F(0)^{k+l-2j} \\
&\quad + \sum_{k,l=0}^{\infty} \sum_{j=0}^{\min(k,l)} \frac{(-1)^{k+l}}{(2j+1)!2^{k+l-2j}(k-j)!(l-j)!} D_F(x-y)^{2j+1} D_F(0)^{l+k-2j}.
\end{aligned} \tag{30}$$

For the sum over k and l, it can be reexpressed as a sum over $p = k+l$ and k,

$$\begin{aligned}
&\langle 0|Te^{i\phi(x)}e^{-i\phi(y)}|0\rangle \\
&\quad = \sum_{p=0}^{\infty} \sum_{k=0}^{p} \sum_{j=0}^{\min(k,p-k)} \frac{(-1)^{p}}{(2j)!(2!)^{p-2j}(k-j)!(p-k-j)!} D_F(x-y)^{2j} D_F(0)^{p-2j} \\
&\quad + \sum_{p=0}^{\infty} \sum_{k=0}^{p} \sum_{j=0}^{\min(k,p-k)} \frac{(-1)^{p}}{(2j+1)!(2!)^{p-2j}(k-j)!(p-k-j)!} D_F(x-y)^{2j+1} D_F(0)^{p-2j}
\end{aligned} \tag{31}$$

We need to further separate different cases. In the first line in the r.h.s. of Eq. (31), when $p = 2q$ is even,

$$\sum_{k=0}^{p} \sum_{j=0}^{\min(k,p-k)} = \sum_{j=0}^{q} \sum_{k=j}^{2q-j} = \sum_{j=0}^{q} \sum_{m=0}^{2q-2j} . \tag{32}$$

At the last step, we defined $m = k - j$. Therefore, the sum is

$$\begin{aligned}
&\sum_{k=0}^{p} \sum_{j=0}^{\min(k,p-k)} \frac{(-1)^p}{(2j)!(2!)^{p-2j}(k-j)!(p-k-j)!} D_F(x-y)^{2j} D_F(0)^{p-2j} \\
&= \sum_{j=0}^{q} \sum_{m=0}^{2q-2j} \frac{1}{(2j)! 2^{2q-2j} m!(2q-2j-m)!} D_F(x-y)^{2j} D_F(0)^{2q-2j} \\
&= \sum_{j=0}^{q} \frac{1}{(2j)! 2^{2q-2j}(2q-2j)!} D_F(x-y)^{2j} D_F(0)^{2q-2j} \sum_{m=0}^{2q-2j} {}_{2q-2j}C_m \\
&= \sum_{j=0}^{q} \frac{1}{(2j)!(2q-2j)!} D_F(x-y)^{2j} D_F(0)^{2q-2j}.
\end{aligned} \tag{33}$$

Similarly for the last line in Eq. (31),

$$\begin{aligned}
&\sum_{k=0}^{p} \sum_{j=0}^{\min(k,p-k)} \frac{(-1)^p}{(2j+1)!(2!)^{p-2j}(k-j)!(p-k-j)!} D_F(x-y)^{2j+1} D_F(0)^{p-2j} \\
&= \sum_{j=0}^{q} \sum_{m=0}^{2q-2j} \frac{1}{(2j+1)! 2^{2q-2j} m!(2q-2j-m)!} D_F(x-y)^{2j+1} D_F(0)^{2q-2j} \\
&= \sum_{j=0}^{q} \frac{1}{(2j+1)! 2^{2q-2j}(2q-2j)!} D_F(x-y)^{2j+1} D_F(0)^{2q-2j} \sum_{m=0}^{2q-2j} {}_{2q-2j}C_m \\
&= \sum_{j=0}^{q} \frac{1}{(2j+1)!(2q-2j)!} D_F(x-y)^{2j+1} D_F(0)^{2q-2j}.
\end{aligned} \tag{34}$$

On the other hand, when $p = 2q + 1$ is odd,

$$\sum_{k=0}^{p} \sum_{j=0}^{\min(k,p-k)} = \sum_{j=0}^{q} \sum_{k=j}^{2q+1-j} = \sum_{j=0}^{q} \sum_{m=0}^{2q+1-2j} . \tag{35}$$

At the last step, we again defined $m = k - j$. Therefore, the sum is

$$\begin{aligned}
&\sum_{k=0}^{p} \sum_{j=0}^{\min(k,p-k)} \frac{(-1)^p}{(2j)!(2!)^{p-2j}(k-j)!(p-k-j)!} D_F(x-y)^{2j} D_F(0)^{p-2j} \\
&= \sum_{j=0}^{q} \sum_{m=0}^{2q+1-2j} \frac{-1}{(2j)! 2^{2q+1-2j} m!(2q+1-2j-m)!} D_F(x-y)^{2j} D_F(0)^{2q+1-2j}
\end{aligned}$$

$$= \sum_{j=0}^{q} \frac{-1}{(2j)!2^{2q+1-2j}(2q+1-2j)!} D_F(x-y)^{2j} D_F(0)^{2q+1-2j} \sum_{m=0}^{2q+1-2j} {}_{2q+1-2j}C_m$$

$$= \sum_{j=0}^{q} \frac{-1}{(2j)!(2q+1-2j)!} D_F(x-y)^{2j} D_F(0)^{2q+1-2j}. \tag{36}$$

Similarly for the last line in Eq. (31),

$$\sum_{k=0}^{p} \sum_{j=0}^{\min(k,p-k)} \frac{(-1)^p}{(2j+1)!(2!)^{p-2j}(k-j)!(p-k-j)!} D_F(x-y)^{2j+1} D_F(0)^{p-2j}$$

$$= \sum_{j=0}^{q} \sum_{m=0}^{2q+1-2j} \frac{-1}{(2j+1)!2^{2q+1-2j} m!(2q+1-2j-m)!} D_F(x-y)^{2j+1} D_F(0)^{2q+1-2j}$$

$$= \sum_{j=0}^{q} \frac{-1}{(2j+1)!2^{2q+1-2j}(2q+1-2j)!} D_F(x-y)^{2j+1} D_F(0)^{2q+1-2j} \sum_{m=0}^{2q+1-2j} {}_{2q+1-2j}C_m$$

$$= \sum_{j=0}^{q} \frac{-1}{(2j+1)!(2q+1-2j)!} D_F(x-y)^{2j+1} D_F(0)^{2q+1-2j}. \tag{37}$$

Putting all four contributions together, Eq. (31) is

$$\langle 0|Te^{i\phi(x)}e^{-i\phi(y)}|0\rangle$$
$$= \sum_{q=0}^{\infty} \sum_{j=0}^{q} \left[\frac{D_F(x-y)^{2j} D_F(0)^{2q-2j}}{(2j)!(2q-2j)!} + \frac{D_F(x-y)^{2j+1} D_F(0)^{2q-2j}}{(2j+1)!(2q-2j)!} - \frac{D_F(x-y)^{2j} D_F(0)^{2q+1-2j}}{(2j)!(2q+1-2j)!} - \frac{D_F(x-y)^{2j+1} D_F(0)^{2q+1-2j}}{(2j+1)!(2q+1-2j)!} \right]. \tag{38}$$

In Eq. (38), the second and third terms give

$$\sum_{j=0}^{q} \left[\frac{D_F(x-y)^{2j+1} D_F(0)^{2q-2j}}{(2j+1)!(2q-2j)!} - \frac{D_F(x-y)^{2j} D_F(0)^{2q+1-2j}}{(2j)!(2q+1-2j)!} \right]$$

$$= \sum_{2j+1=1}^{2q+1} \frac{D_F(x-y)^{2j+1} D_F(0)^{2q+1-(2j+1)}}{(2j+1)!(2q+1-(2j+1))!} - \sum_{2j=0}^{2q} \frac{D_F(x-y)^{2j} D_F(0)^{2q+1-2j}}{(2j)!(2q+1-2j)!}$$

$$= \sum_{n=0}^{2q+1} \frac{(-1)^{2q+1-n} D_F(x-y)^n D_F(0)^{2q+1-n}}{n!(2q+1-n)!}$$

$$= \frac{1}{(2q+1)!} (D_F(x-y) - D_F(0))^{2q+1}. \tag{39}$$

The first term in Eq. (38) has the total power in propagator of $2q$, while the fourth term $2q+2$. Therefore, we separate out the term $q=0$ in the first term, which is 1. Then we rewrite the rest with $q \to q+1$,

$$\sum_{j=0}^{q+1} \frac{D_F(x-y)^{2j} D_F(0)^{2q+2-2j}}{(2j)!(2q+2-2j)!} - \sum_{j=0}^{q} \frac{D_F(x-y)^{2j+1} D_F(0)^{2q+1-2j}}{(2j+1)!(2q+1-2j)!}$$
$$= \sum_{n=0}^{2q+2} \frac{(-1)^{2q+2-n} D_F(x-y)^n D_F(0)^{2q+2-n}}{n!(2q+2-n)!}$$
$$= \frac{1}{(2q+2)!}(D_F(x-y) - D_F(0))^{2q+2}. \tag{40}$$

Adding all contributions, we now find Eq. (38) is

$$\langle 0|Te^{i\phi(x)}e^{-i\phi(y)}|0\rangle$$
$$= 1 + \sum_{q=0}^{\infty} \left[\frac{1}{(2q+1)!}(D_F(x-y) - D_F(0))^{2q+1} + \frac{1}{(2q+2)!}(D_F(x-y) - D_F(0))^{2q+2} \right]$$
$$= e^{D_F(x-y) - D_F(0)} \tag{41}$$

as desired.

(b)

Because of the symmetry requirement under the transformation $\phi(x) \to \phi(x) - \alpha$, the only way to write an invariant that does not change under a constant shift is to take the derivative

$$\vec{\nabla}\phi(x) \to \vec{\nabla}(\phi(x) - \alpha) = \vec{\nabla}\phi(x). \tag{42}$$

Because of the rotational symmetry of space, the derivatives need to be combined in the form $(\vec{\nabla}\phi)^2$. In two spatial dimensions, a possible alternative is $\epsilon_{ij}\nabla_i\phi\nabla_j\phi$, but this vanishes due to the anti-symmetry. Higher order invariants under rotation include $[(\vec{\nabla}\phi)^2]^2$ etc, but they are non-renormalizable. Therefore, the only possible renormalizable Lagrangian under the global symmetry of spin and rotation of space is the form given in the problem, namely

$$\int d^D x \frac{1}{2}\rho(\vec{\nabla}\phi)^2. \tag{43}$$

(c)

The field $s(x) = Ae^{i\phi(x)}$ is the density of spins, and the natural size is given by $A = \xi\hbar/a^D$ where a is the lattice constant and $\xi = O(1)$ is a numerical dimensionless constant of order unity. (I set $\hbar = 1$ below.) On the other hand, the field ϕ is in the exponent and hence must be dimensionless, and hence the dimension of the spin wave modulus ρ is L^{-D+2}. We expect it to be proportional to $a^{-(D-2)}$. In the case of the Heisenberg model, we have found $\rho = j^2\beta J/a^{D-2}$.

The Feynman propagator for a massless scalar field in Euclidean D dimensions with the canonical kinetic term

$$\int d^D x \frac{1}{2}(\vec{\nabla}\phi)^2 \tag{44}$$

is worked out most easily with the Feynman parameter

$$\begin{aligned}
D_F(x) &= \int \frac{d^D p}{(2\pi)^D}\frac{e^{-ip\cdot x}}{p^2} \\
&= \int \frac{d^D p}{(2\pi)^D}\int_0^\infty dt\ e^{-tp^2}e^{-ip\cdot x} \\
&= \int \frac{d^D p}{(2\pi)^D}\int_0^\infty dt\ e^{-t(p+ix/2t)^2-x^2/4t} \\
&= \frac{1}{(2\pi)^D}\int_0^\infty dt \left(\frac{\pi}{t}\right)^{D/2} e^{-x^2/4t} \\
&= \frac{1}{(4\pi)^{D/2}}\int_0^\infty \frac{dt}{t^{D/2}} e^{-x^2/4t} \\
&= \frac{1}{(4\pi)^{D/2}}\int_0^\infty ds\ s^{\frac{D}{2}-2} e^{-sx^2/4} \\
&= \frac{1}{(4\pi)^{D/2}}\left(\frac{4}{x^2}\right)^{\frac{D}{2}-1}\Gamma\left(\frac{D}{2}-1\right) \\
&= \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{D/2}}\frac{1}{(x^2)^{\frac{D}{2}-1}}\ .
\end{aligned} \tag{45}$$

In the sixth line, we changed the integration variable $t = 1/s$.

For $D = 2$, one needs to "renormalize" the propagator. Using $D = 2+2\epsilon$,

$$\begin{aligned}
D_F(x) &= \frac{\Gamma(\epsilon)}{4\pi}\frac{1}{(\pi x^2)^\epsilon} \\
&= \frac{1}{4\pi}\left(\frac{1}{\epsilon}-\gamma\right)\left(1-\epsilon\ln\pi x^2\right) \\
&= \text{constant} - \frac{1}{4\pi}\ln x^2.
\end{aligned} \tag{46}$$

Note that the divergent constant is common for $D_F(x)$ and $D_F(0)$ and hence cancels when we compute the spin-spin correlation function.

In our case, the kinetic term is not canonical $\propto \rho$ and hence the two-point function is changed $\propto \rho^{-1}$,

$$D_F(x) = \frac{1}{\rho}\frac{\Gamma(\frac{D}{2}-1)}{4\pi^{D/2}}\frac{1}{(x^2)^{\frac{D}{2}-1}} \quad (D \neq 2), \qquad -\frac{1}{\rho}\frac{1}{4\pi}\ln x^2 \quad (D=2). \quad (47)$$

In the expression for the spin-spin correlation function, we need to evaluate the propagator at the origin $D_F(0)$. The natural cutoff is again the atomic scale and we instead use $D_F(a)$.

We now discuss each dimensions separately. In $D > 2$,

$$\langle s(0)s^*(x)\rangle = \frac{c^2}{a^{2D}} \exp \frac{1}{\rho}\frac{\Gamma(\frac{D}{2}-1)}{4\pi^{D/2}} \left[(x^2)^{-(\frac{D}{2}-1)} - a^{-(D-2)}\right]. \quad (48)$$

The second term in the exponent is order unity and gives an overall numerical constant. As the spins are separated, the x dependence disappears quickly as power in the exponent, and the correlation function asymptotes to a constant,

$$\langle s(0)s^*(x)\rangle \xrightarrow{x^2\to\infty} \frac{c^2}{a^{2D}} \exp \frac{1}{\rho a^{D-2}}\frac{\Gamma(\frac{D}{2}-1)}{4\pi^{D/2}} \neq 0 \quad (49)$$

signaling the order parameter of spontaneous symmetry breaking of the $SO(2)$ symmetry.

In $D = 2$,

$$\begin{aligned}\langle s(0)s^*(x)\rangle &= \frac{c^2}{a^4} \exp -\frac{1}{\rho}\frac{1}{4\pi}\left[\ln x^2 - \ln a^2\right] \\ &= \frac{c^2}{a^4}\left(\frac{x^2}{a^2}\right)^{-1/(4\pi\rho)}. \end{aligned} \quad (50)$$

Therefore the correlation function damps as power and hence there is no true long-range order of spontaneous symmetry breaking of the $SO(2)$ symmetry. This behavior is consistent with the Mermin–Wagner theorem. Another curious fact is that the critical exponent of the correlation function

$$\eta = \frac{1}{2\pi\rho} \quad (51)$$

varies continuously. This case is basically that there is a continuum of critical points, a very special behavior in two dimensions.

In $D = 1$, the propagator is

$$D_F(x) = \frac{1}{\rho}\frac{\Gamma(-\frac{1}{2})}{4\pi^{1/2}}|x| = -\frac{|x|}{2\rho}\ . \tag{52}$$

Therefore the spin-spin correlation function

$$\langle s(0)s^*(x)\rangle = \frac{c^2}{a^2}e^{-|x|/2\rho} \tag{53}$$

damps exponentially. Again, there is no long-range order but there is a finite correlation length $\xi = 2\rho$.

Here is an apparent paradox. Under the $U(1) \simeq SO(2)$ symmetry, $s(x) \to s(x)e^{i\alpha}$, and hence $\phi \to \phi + \alpha$. The correlation function $\langle s(0)s^*(x)\rangle$ is invariant under this symmetry and hence can be (and is) non-zero. On the other hand, $\langle s(0)s(x)\rangle$ is not invariant and hence must vanish. However, an explicit calculation shows it does not vanish. If you go through the same calculation as in part (a), we can compute $\langle 0|Te^{i\phi(y)}e^{i\phi(z)}|0\rangle$ by using the source $J(x) = \delta^D(x-y) + \delta^D(x-z)$. The result is then simply $\langle 0|Te^{i\phi(y)}e^{i\phi(z)}|0\rangle = e^{D_F(y-z)+D(0)} \neq 0$. How can this be consistent with the argument based on symmetry above that it should vanish?

The point is that the path integral needs to be done more carefully when the action does not depend on the constant piece ("zero mode") of the field, namely $m = 0$ for a free field. Write $\phi(x) = \phi_0 + \phi'(x)$, where ϕ' does not contain the zero mode $\int d^Dx\phi'(x) = 0$. The path integral Eq. (3) can then be written as

$$\begin{aligned}\langle 0|Te^{i\phi(y)}e^{-i\phi(z)}|0\rangle &= \frac{1}{Z}\int \mathcal{D}\phi(x)e^{i\int d^Dx[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi]}e^{i\phi(y)}e^{-i\phi(z)} \\ &= \frac{1}{Z}\int d\phi_0\mathcal{D}\phi'(x)e^{i\int d^Dx[\frac{1}{2}\partial_\mu\phi'\partial^\mu\phi']}e^{i(\phi_0+\phi'(y))}e^{-i(\phi_0+\phi'(z))}.\end{aligned} \tag{54}$$

In this case, ϕ_0 cancels and the integrand does not depend on ϕ_0 at all. Hence $\int_0^{2\pi} d\phi_0 = 2\pi =$ constant and the computation is the same as before. On the other hand,

$$\begin{aligned}\langle 0|Te^{i\phi(y)}e^{i\phi(z)}|0\rangle &= \frac{1}{Z}\int \mathcal{D}\phi(x)e^{i\int d^Dx[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi]}e^{i\phi(y)}e^{i\phi(z)} \\ &= \frac{1}{Z}\int d\phi_0\mathcal{D}\phi'(x)e^{i\int d^Dx[\frac{1}{2}\partial_\mu\phi'\partial^\mu\phi']}e^{i(\phi_0+\phi'(y))}e^{i(\phi_0+\phi'(z))}.\end{aligned} \tag{55}$$

The ϕ_0 dependence is $e^{2i\phi_0}$ overall, and its integral vanishes,

$$\int_0^{2\pi} d\phi_0\ e^{2i\phi_0} = 0. \tag{56}$$

Therefore, the correlation function $\langle s(0)s(x)\rangle$ indeed vanishes consistent with the symmetry.

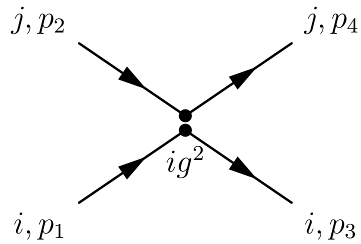

Figure 1: The tree-level Feynman diagram for the fermion four-point function.

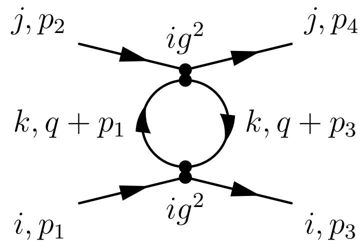

Figure 2: The one-loop t-channel Feynman diagram for the fermion four-point function enhanced at large N. Because of the closed fermion loop, it comes with an overall minus sign in addition to the Feynman rules.

Problem 12.2

We compute the beta function of the Gross–Neveu model using perturbation theory.

$$\mathcal{L} = \sum_i^N \bar{\psi}_i i \partial\!\!\!/ \psi_i + \frac{1}{2} g_0^2 \left(\sum_i^N \bar{\psi}_i \psi_i \right)^2 \tag{57}$$

for $i = 1, \cdots, N$. Note that the sum over i is taken inside the parenthes before taking the square, which is not very clear in the way Peskin–Schroeder writes the Lagrangian density. I've put the subscript g_0 to indicate that it is the bare coupling.

We compute the 1PI four-point function $\psi_i(p_1)\psi_j(p_2) \to \psi_i(p_3)\psi_j(p_4)$, specifically for $i \neq j$ to avoid complications with double counting etc. The tree-level Feynman rule is simply $i\Gamma_0^{(4)} = ig_0^2$. At the one-loop level, we first consider only the t-channel diagram which is enhanced by N. We will come back to the diagrams suppressed by $1/N$ later on. The one-loop 1PI four-point function is then

$$i\Gamma_{1t1}^{(4)} = (-1)(ig_0^2)^2 N \int \frac{d^D q}{(2\pi)^D} \mathrm{Tr} \frac{i}{q\!\!\!/ + p\!\!\!/_1} \frac{i}{q\!\!\!/ + p\!\!\!/_3}$$

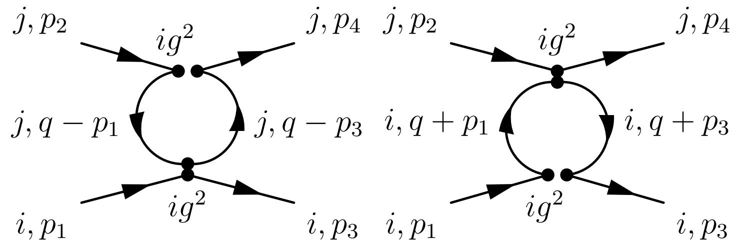

Figure 3: The one-loop t-channel Feynman diagrams for the fermion four-point function not enhanced by large N. Because the fermion loop is not closed, there is not an overall minus sign in addition to the Feynman rules.

$$= -2Ng_0^4 \int_0^1 dz \int \frac{d^D q}{(2\pi)^D} \left[\frac{1}{q^2 + z(1-z)t} - \frac{3z(1-z)t/2}{(q^2+z(1-z)t)^2} \right]. \quad (58)$$

Here, $t = (p_1 - p_3)^2 = -2p_1 \cdot p_3$. Now we compute it with various different renormalization schemes. I hope this simple exposition in different renormalization schemes would tell you their differences and equivalence. Note that the on-shell renormalization or zero-momentum subtraction does not work for this calculation because the evaluation of the above integral for $s = 4m^2 = 0$, $t = u = 0$ leads to an infrared singularity. We need to employ other renormalization schemes.

There are two more t-channel diagrams without a closed fermion loop. The amplitude with the fermion type i running in the loop is

$$\begin{aligned} i\Gamma^{(4)}_{1t2} &= (ig_0^2)^2 \int \frac{d^D q}{(2\pi)^D} \frac{i}{\not{q} + \not{p}_3} \frac{i}{\not{q} + \not{p}_1} \\ &= g_0^4 \int_0^1 dz \int \frac{d^D q}{(2\pi)^D} \frac{(\not{q} + z\not{p}_3)(\not{q} + (1-z)\not{p}_1)}{(q^2 + z(1-z)t)^2} \\ &= g_0^4 \int_0^1 dz \int \frac{d^D q}{(2\pi)^D} \frac{q^2 + z(1-z)\not{p}_3\not{p}_1}{(q^2 + z(1-z)t)^2} \\ &= g_0^4 \int_0^1 dz \int \frac{d^D q}{(2\pi)^D} \left[\frac{1}{q^2 + z(1-z)t} - \frac{z(1-z)t}{(q^2+z(1-z)t)^2} \right]. \end{aligned} \quad (59)$$

At the last step, we used the fact that $\not{p}_1 u(p_1) = 0$ and similarly $\bar{u}(p_3)\not{p}_3 = 0$. The same diagram for the species j running in the loop gives exactly the same amplitude.

The s-channel diagram gives the amplitude

$$i\Gamma^{(4)}_{1s} = (ig_0^2)^2 N \int \frac{d^D q}{(2\pi)^D} \frac{i}{-\not{q} + \not{p}_1} \frac{i}{\not{q} + \not{p}_2} \quad (60)$$

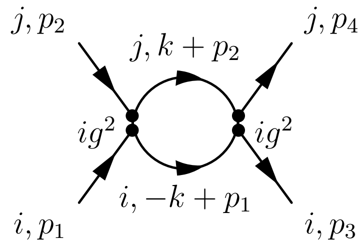

Figure 4: The one-loop s-channel Feynman diagram for the fermion four-point function. Because the fermion loop is not closed, there is not an overall minus sign in addition to the Feynman rules.

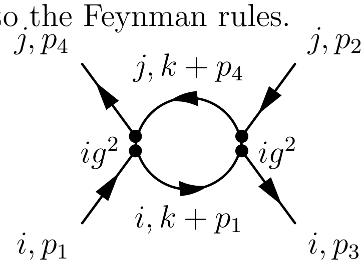

Figure 5: The one-loop u-channel Feynman diagram for the fermion four-point function. Because the fermion loop is not closed, there is not an overall minus sign in addition to the Feynman rules.

Note that two propagators are not multiplied because each of them comes with external fermion lines. Similarly, the u-channel diagram gives

$$i\Gamma^{(4)}_{1u} = (ig_0^2)^2 N \int \frac{d^D q}{(2\pi)^D} \frac{i}{\not{q} + \not{p}_1} \frac{i}{\not{q} + \not{p}_4} \tag{61}$$

Both s- and u-channel diagrams are logarithmically divergent, but the divergent pieces cancel between them. The remaining finite piece does not contribute to the beta function and we will not consider them further below.

(a) Bare Perturbation Theory with Momentum Cutoff

This is what is closest to the Wilsonian approach of integrating out momentum slices. We first do the Wick rotation to the Euclidean space and the one-loop amplitude is

$$\begin{aligned} i\Gamma^{(4)}_{1t1} &= -2iNg_0^4 \int_0^1 dz \int \frac{d^2 q_E}{(2\pi)^D} \left[\frac{1}{-q_E^2 + z(1-z)t} - \frac{3z(1-z)t/2}{(-q_E^2 + z(1-z)t)^2} \right] \\ &= -2iNg_0^4 \int_0^1 dz \int_0^{\Lambda^2} \frac{dq_E^2}{4\pi} \left[-\frac{1}{q_E^2 - z(1-z)(-t)} - \frac{3z(1-z)(-t)/2}{(q_E^2 + z(1-z)(-t))^2} \right] \end{aligned}$$

$$= -2iNg_0^4 \int_0^1 dz \frac{1}{4\pi} \left[-\ln[q_E^2 + z(1-z)(-t)] - \frac{3z(1-z)(-t)/2}{q_E^2 + z(1-z)(-t)} \right]_0^{\Lambda^2}$$
$$= -2iNg_0^4 \int_0^1 dz \left[-\ln \frac{\Lambda^2}{z(1-z)(-t)} + \frac{3}{2} + O\left(\frac{-t}{\Lambda^2}\right) \right]$$
$$= 2iNg_0^4 \ln \frac{\Lambda^2}{-t} - \frac{1}{2} + O\left(\frac{-t}{\Lambda^2}\right). \tag{62}$$

The other t-channel amplitudes $i\Gamma^{(4)}_{1t2}$ for species i and j running in the loop give the similar contribution without a factor of N with the opposite sign. 1PI four-point function is then

$$\Gamma^{(4)}(p_1, p_2, p_3, p_4) = g_0^2 + \frac{(N-2)g_0^4}{2\pi} \ln \frac{\Lambda^2}{-t} + \Lambda\text{-independent} + O\left(\frac{-t}{\Lambda^2}\right) + O(g_0^6), \tag{63}$$

where we ignored the s- and u-channel diagrams without divergence (and hence logarithm).

In the Wilsonian approach, we keep integrating out a slice in the momentum space. The path integral over the momentum slice gives corrections to the action in a way to modify the coupling constant and other parameters. The physics of course is left intact because it is just doing the path integral step by step instead of all at once. What it mounts to is that the physica quantity, such as the 1PI four-point function above, does not depend on the cutoff as long as the coupling constant g_0 is changed accordingly. The requirement is therefore (Callan–Symanzik equation)

$$\Lambda \frac{D}{D\Lambda} \Gamma^{(4)}(\{p_i\}; \Lambda, g_0) = \left[\Lambda \frac{\partial}{\partial \Lambda} + \Lambda \frac{\partial g_0}{\partial \Lambda} \frac{\partial}{\partial g_0} \right] \Gamma^{(4)}(\{p_i\}; \Lambda, g_0) = 0. \tag{64}$$

Here, we used the fact that there is no wave function renormalization at this order in perturbation theory. Substituting the explicit result into the Callan–Symanzik equation, we find

$$\beta(g_0) = \Lambda \frac{\partial g_0}{\partial \Lambda} = -\frac{1}{2\pi}(N-2)g_0^3. \tag{65}$$

This is the renormalization-group equation (or evolution equation) of the coupling constant.

Now that we have the beta function, we can integrate it and work out the cutoff dependence of the coupling constant explicitly. We first write it as

$$\Lambda \frac{\partial}{\partial \Lambda} \frac{1}{g_0^2} = \frac{N-2}{\pi}, \tag{66}$$

and hence

$$(N-2)g_0^2(\Lambda') = \frac{1}{((N-2)g_0^2)^{-1}(\Lambda) - \frac{1}{\pi}\ln\frac{\Lambda}{\Lambda'}}. \tag{67}$$

At the cutoff $\Lambda' = \Lambda e^{-\pi/(N-2)g_0^2(\Lambda)}$ the coupling constant becomes infinite.

Note that the solution to the renormalization group equation is not the same as the perturbative result, since

$$\begin{aligned}(N-2)g_0^2(\Lambda') = (N-2)g_0^2(\Lambda) \times \Bigg[1 + \frac{(N-2)g_0^2(\Lambda)}{\pi}\ln\frac{\Lambda}{\Lambda'} \\ + \left(\frac{(N-2)g_0^2(\Lambda)}{\pi}\ln\frac{\Lambda}{\Lambda'}\right)^2 + \left(\frac{(N-2)g_0^2(\Lambda)}{\pi}\ln\frac{\Lambda}{\Lambda'}\right)^3 + \cdots\Bigg].\end{aligned} \tag{68}$$

Therefore, the solution is formally an all-order result. The point here is this. At each step in Wilsonian integration over a momentum slice, the logarithm is small, and the change in the coupling constant is small. Therefore, perturbation theory can be trusted for the infinitesimal step. "All-at-one quantization" tries to integrate over a large range in momentum space, resulting in a large logarithm that makes the perturbation theory unreliable. If we study the infinitesimal slice and sum it up in the form of a differential equation, the solution is the accumulation of trustworthy small corrections and is a reliable calculation. In this way, differentiating the 1PI function and then integrate it will give you a much more reliable answer than the original calculation.

In general, the one-loop calculation produces terms of the type

$$\frac{g^2}{\pi}\ln\frac{\Lambda}{\Lambda'}\,, \qquad \frac{g^2}{\pi}\,. \tag{69}$$

If you integrate the beta-function at the one-loop level, it sums all orders of terms

$$\left(\frac{g^2}{\pi}\right)^n \ln^n\frac{\Lambda}{\Lambda'}\,. \tag{70}$$

This result is called "leading-log approximation," since it sums all terms enhanced by the highest powers of the logarithms when the logarithm is large.

On the other hand, the two-loop calculation yields terms of the type

$$\left(\frac{g^2}{\pi}\right)^2 \ln^2\frac{\Lambda}{\Lambda'}\,, \qquad \left(\frac{g^2}{\pi}\right)^2 \ln\frac{\Lambda}{\Lambda'}\,, \qquad \left(\frac{g^2}{\pi}\right)^2\,. \tag{71}$$

The first term is actually already taken care of by the integrated one-loop beta function and hence is not new. The two-loop contribution to the beta function gives all terms of the type

$$\left(\frac{g^2}{\pi}\right)^{n+1} \ln^n\frac{\Lambda}{\Lambda'}\,. \tag{72}$$

perturbation	L^0	L^1	L^2	L^3	L^4
tree	tree				
1-loop	1-loop	1-loop+RGE			
2-loop	2-loop	2-loop+RGE	1-loop+RGE		
3-loop	3-loop	3-loop+RGE	2-loop+RGE	1-loop+RGE	
4-loop	4-loop	4-loop+RGE	3-loop+RGE	2-loop+RGE	1-loop+RGE

Table 1: What terms arise in perturbation theory and how they are resummed in renormalization-group equations. Here, L is a logarithm.

In general, the k-loop perturbation theory gives terms of the type

$$\left(\frac{g^2}{\pi}\right)^k \ln^l \frac{\Lambda}{\Lambda'}\ , \qquad (l = k, k-1, \cdots, 0). \tag{73}$$

The renormalization equation with k-loop beta function, once integrated, sums all contributions of the type

$$\left(\frac{g^2}{\pi}\right)^{n+k} \ln^n \frac{\Lambda}{\Lambda'}\ . \tag{74}$$

In this way, the renormalization group equation sums perturbation series in different orders, giving more reliable results than the simple perturbation theoy when the logarithm becomes large. (In $D = 4$, π is replaced by $4\pi^2$.)

(b) $\overline{MS}$

With dimensional regularization, remember that the bare coupling g_0^2 has a slight mass dimension. In $D = 2 - 2\epsilon$ dimensions, the mass dimension of the bare coupling is 2ϵ. We write therefore $g_0^2 = g^2\mu^{2\epsilon}$ apart from the counter terms which we will determine using the $\overline{MS}$ scheme. At this point, μ is an arbitrary parameter of mass dimension introduced to fix up the dimension of the coupling constant. Within the $\overline{MS}$ scheme, it turns out to have the meaning of the renormalization scale. We only consider the logarithmically divergent piece.

$$\begin{aligned}
i\Gamma_1^{(4)} &= -2(N-2)g^4\mu^{4\epsilon}\int_0^1 dz \int \frac{d^Dq}{(2\pi)^D}\left[\frac{1}{q^2 + z(1-z)t}\right] \\
&= 2(N-2)g^4\mu^{4\epsilon}\frac{i}{(4\pi)^{1-\epsilon}}\int_0^1 dz\Gamma(\epsilon)\left(z(1-z)(-t)\right)^{-\epsilon} \\
&= 2(N-2)g^4\mu^{4\epsilon}\frac{i}{(4\pi)^{1-\epsilon}}\Gamma(\epsilon)\frac{\Gamma(1-\epsilon)\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}(-t)^{-\epsilon}.
\end{aligned} \tag{75}$$

At the last step, we used the identity

$$B(p,q) = \int_0^1 dz\ z^{p-1}(1-z)^{q-1} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\ . \tag{76}$$

Expanding it up to $O(\epsilon^0)$, we obtain

$$i\Gamma_1^{(4)} \quad = \quad i(N-2)g^4\mu^{4\epsilon}\frac{1}{2\pi}(-t)^{-\epsilon}\left(\frac{1}{\epsilon} - \gamma + \ln 4\pi\right). \tag{77}$$

t dependence is not expanded yet to make the whole expression have the same dimension as the tree-level amplitude $i\Gamma_0^{(4)} = ig^2\mu^{2\epsilon}$. Again we omitted the s- and u-channel contributions without $O(N)$ enhancement or divergence (hence logarithm). The counter term is determined to cancel the pole $\frac{1}{\epsilon} - \gamma + \ln 4\pi$,[1]

$$i\Gamma_{ct}^{(4)} = -i(N-2)g^4\mu^{2\epsilon}\frac{1}{2\pi}\left(\frac{1}{\epsilon} - \gamma + \ln 4\pi\right). \tag{78}$$

Here, $\mu^{2\epsilon}$ is there to keep the same dimension for the amplitude. Then the sum is finite in the limit $\epsilon \to 0$,

$$i\Gamma_0^{(4)} + i\Gamma_1^{(4)} + i\Gamma_{ct}^{(4)} = ig^2\left(1 - \frac{(N-2)g^2}{2\pi}\ln\frac{-t}{\mu^2}\right). \tag{79}$$

Therefore, g has the meaning of the coupling measured at $t = -\mu^2$, namely that this fictitious parameter μ turns out to the be renormalization scale. The 1PI effective action at this order is hence

$$\Gamma^{(4)}(p_1,p_2,p_3,p_4) = g^2\left(1 - \frac{(N-2)g^2}{2\pi}\ln\frac{-t}{\mu^2}\right) + O(g^6). \tag{80}$$

The Callan–Symanzik equation says

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - 4\gamma\right]\Gamma^{(4)}(p_1,p_2,p_3,p_4) = 0. \tag{81}$$

At this order, $\gamma = 0$. On the other hand,

$$\mu\frac{\partial}{\partial\mu}\Gamma^{(4)} = (N-2)g^4\frac{1}{\pi} + O(g^6) = -\beta(g)(2g + O(g^3)), \tag{82}$$

and hence

$$\beta(g) = -\frac{(N-2)g^3}{2\pi}. \tag{83}$$

[1] The "Minimal Subtraction" scheme cancels only the pole $1/\epsilon$ by the counter term. $\overline{MS}$ (pronounced Em-Es-Bah) is "Modified Minimal Subtraction" scheme that includes $-\gamma + \ln 4\pi$.