HW #6

1. Inflaton

(a) slow-roll regime

In the slow-roll regime, we neglect the kinetic energy as well as $\ddot{\phi}$ term in the equation of motion. Then
$H^2 = \frac{8\pi}{3} G_N \frac{m^2}{2} \phi^2$, $3 H \dot{\phi} + m^2 \phi = 0$.
We use $8 \pi G_N = M_{\rm Pl}{}^{-2}$. Putting them together,
$3 \frac{m \phi}{\sqrt{6} M_{\rm Pl}} \dot{\phi} + m^2 \phi = 0$,
and hence
$d\phi = -\sqrt{\frac{2}{3}} m M_{\rm Pl} dt$.
The solution is simply
$\phi(t) = \phi(0) - \sqrt{\frac{2}{3}} m M_{\rm Pl} t$.
Using this solution, the kinetic energy is
$\frac{1}{2} \dot{\phi}^2 = \frac{1}{3} m^2 M_{\rm Pl}{}^2$,
while the potential energy is
$\frac{m^2}{2} \phi^2$.
Therefore, if $\phi \gg M_{\rm Pl}$, we see that the kinetic energy is indeed smaller than the potential energy. At the same time, $\ddot{\phi} = 0$ for the above solution and indeed is negligible compared other non-vanishg terms. The scale factor can be obtained from the Friedmann equation
$H^2 = \frac{\dot{R}^2}{R^2} = \frac{1}{3 M_{\rm Pl}{}^2} \frac{1}{2} m^2 \phi^2$,
$\frac{\dot{R}}{R} = \frac{1}{\sqrt{6} M_{\rm Pl}} m \phi$,
$d \log R = \frac{1}{\sqrt{6} M_{\rm Pl}} m \phi\, dt$
and hence
$\log \frac{R(t)}{R(0)} = \frac{1}{\sqrt{6} M_{\rm Pl}} m\left(\phi(0) t - \frac{1}{\sqrt{6}} m M_{\rm Pl} t^2\right)$
When $\phi \gg M_{\rm Pl}$, the first term in the parentheses dominates and the scale factor grows exponentially with time.

(b) oscillating regime

In the oscillating regime $\phi \ll M_{\text{Pl}}$ and $m\,t \gg 1$, we study the equation of motion
$\ddot{\phi} + 3\,H\,\dot{\phi} + m^2\,\phi = 0,$
$H^2 = \frac{1}{3\,M_{\text{Pl}}^2}\,\frac{1}{2}\left(\dot{\phi}^2 + m^2\,\phi^2\right)$
with the ansatz
$\phi(t) = \phi(t_1)\,\frac{t_1}{t}\,\cos m(t - t_1).$
$\ddot{\phi} = \phi(t_1)\,\frac{t_1}{t}\,(-m^2\cos m(t - t_1) + 2\,\frac{m}{t}\,\sin m(t - t_1) + \frac{2}{t^2})$
The leading term in $1/(m\,t)$ is the first term in the parantheses which is canceled by $m^2\,\phi$ term. To see if the next-leading term is also cancelled in the equation of motion, we work out
$\dot{\phi} = \phi(t_1)\,\frac{t_1}{t}\,(-m\sin m(t - t_1) - \frac{1}{t}\,\cos m(t - t_1)),$
$H^2 = \frac{1}{3\,M_{\text{Pl}}^2}\,\frac{1}{2}\,\phi(t_1)^2\,\frac{t_1^2}{t^2}\,(m^2 + 2\,\frac{m}{t}\,\sin m(t - t_1)\cos m(t - t_1) + \frac{1}{t^2}\,\cos^2 m(t - t_1)).$
The equation of motion is then
$\phi(t_1)\,\frac{t_1}{t}\,(2\,\frac{m}{t}\,\sin m(t - t_1) + \frac{2}{t^2}) +$

$$3\,\frac{1}{\sqrt{6}\,M_{\text{Pl}}}\,\frac{t_1}{t}\,\sqrt{m^2 + 2\,\frac{m}{t}\,\sin m(t - t_1)\cos m(t - t_1) + \frac{1}{t^2}\,\cos^2 m(t - t_1)}\;\phi(t_1)^2\,\frac{t_1}{t}\,(-m\sin m(t - t_1) - \frac{1}{t}\,\cos m(t - t_1)) = 0.$$

The next leading terms are
$\phi(t_1)\,\frac{t_1}{t}\,(2\,\frac{m}{t}\,\sin m(t - t_1)) - 3\,\frac{1}{\sqrt{6}\,M_{\text{Pl}}}\,\frac{t_1}{t}\,m\,\phi(t_1)^2\,\frac{t_1}{t}\,m\sin m(t - t_1) = 0.$
This is satisfied if
$2 - 3\,\frac{1}{\sqrt{6}\,M_{\text{Pl}}}\,t_1\,m\,\phi(t_1) = 0.$
$t_1 = 2\sqrt{\frac{2}{3}}\,\frac{M_{\text{Pl}}}{m\,\phi(t_1)}.$
With this choice, the equation of motion is satisfied both for the leading and the next-leading terms in the expansion in $1/m\,t$.

The expansion rate to the leading order is
$H^2 = \frac{1}{3\,M_{\text{Pl}}^2}\,\frac{1}{2}\,\phi(t_1)^2\,\frac{t_1^2}{t^2}\,m^2 = \frac{1}{3\,M_{\text{Pl}}^2}\,\frac{1}{2}\,4\,\frac{2}{3}\,\frac{M_{\text{Pl}}^2}{m^2}\,\frac{1}{t^2}\,m^2 = \frac{4}{9}\,\frac{1}{t^2}$
This differential equation can be integrated easily and we find
$R = R(t_1)\left(\frac{t}{t_1}\right)^{2/3}.$
Therefore the energy density scales as
$\rho = 3\,M_{\text{Pl}}^2\,H^2 = 3\,M_{\text{Pl}}^2\,\frac{4}{9}\,\frac{1}{t^2} = \frac{4}{3}\,\frac{1}{t_1^2}\,(\frac{R(t_1)}{R})^3$
Indeed, the energy density is that of matter-domnated universe.

(c) numerical solution

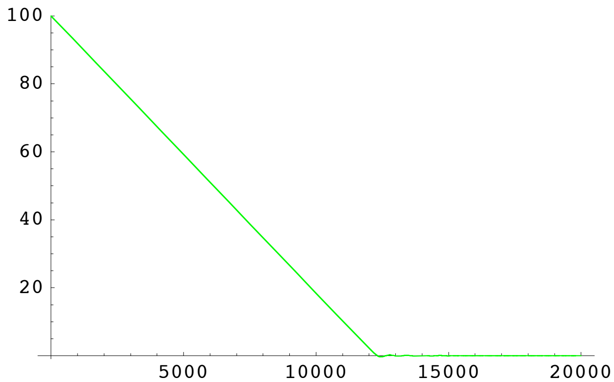
We solve the differential equation numerically. A word of caution is that this is actually a not *very* safe thing to do if the solution oscillates crazy like this one. The numerical errors may build up. *Mathematica* seems to handle it fairly well, though.

I didn't specify the boundary conditions. Normally, we take the initial time derivative to vanish, but of course we don't really know what the *right* boundary conditions are.

```
sol = NDSolve[{ϕ''[t] + 3 H ϕ'[t] + m^2 ϕ[t] == 0 /. {H → Sqrt[1/(3 MPl^2) 1/2 (ϕ'[t]^2 + m^2 ϕ[t]^2)]} /.
    {MPl → 1, m → 10^-2}, ϕ[0] == 100, ϕ'[0] == 0}, ϕ, {t, 0, 100000}]
```

```
{{ϕ → InterpolatingFunction[{{0., 100000.}}, <>]}}
```

```
Plot[Evaluate[ϕ[t] /. sol[[1]]], {t, 0, 20000},
 PlotRange → {-1, 100}, PlotStyle → RGBColor[0, 1, 0]]
```

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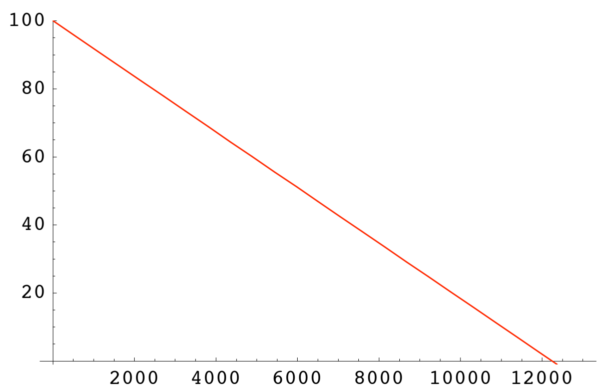
The slow-roll solution does not have vanishing time derivative at the initial time.

```
Plot[ϕ0 - Sqrt[2/3] m MPl t /. {ϕ0 → 100, m → 10^-2, MPl → 1},
 {t, 0, 13000}, PlotRange → {-1, 100}, PlotStyle → RGBColor[1, 0, 0]]
```

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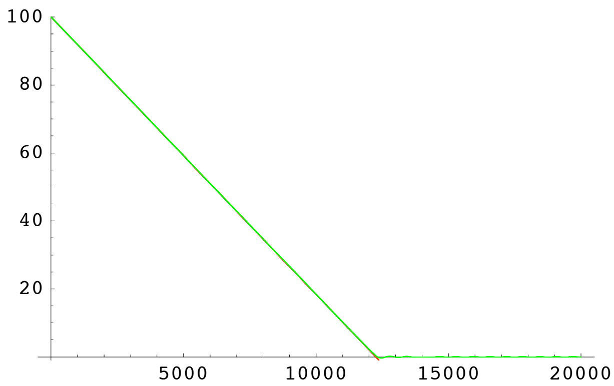

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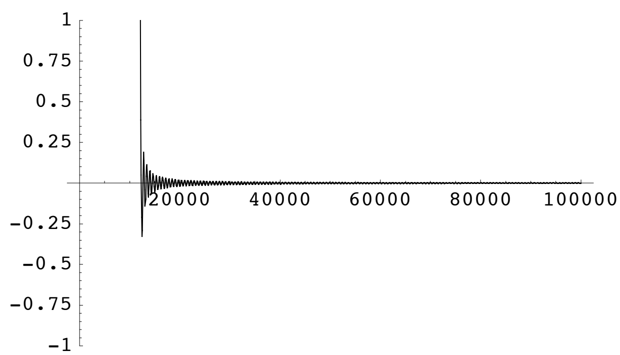
The fact that they agree so well with each other is a demonstration that the inflation is quite insensitive to the initial values; the solution quickly approaches the slow-roll solution.

```
Plot[Evaluate[ϕ[t] /. sol[[1]]], {t, 10000, 100000}, PlotRange → {-1, 1}]
```



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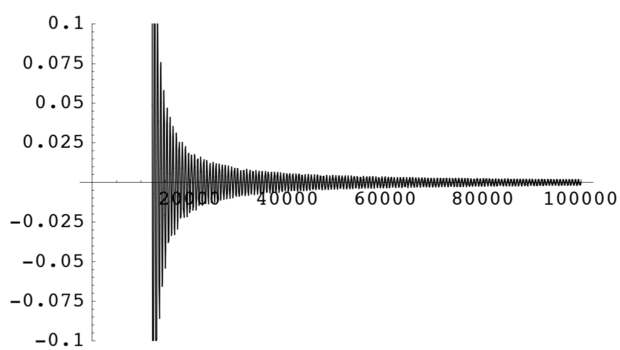
```
Plot[Evaluate[ϕ[t] /. sol[[1]]], {t, 10000, 100000}, PlotRange → {-0.1, 0.1}]
```



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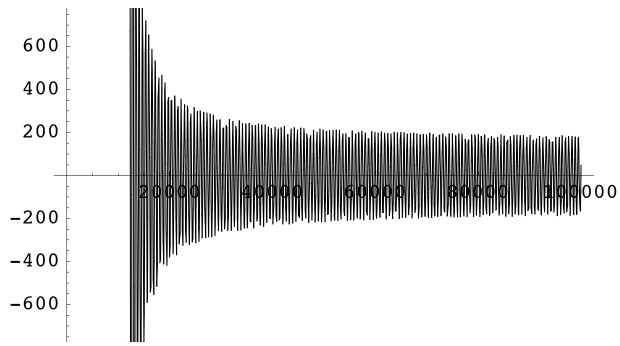
Note that the time t in this solution is not the same as the time t in the analytic solution in the oscillating regime; they are offset by the contribution from the slow-roll regime. But the offset is quickly forgotten as time goes on $m\,t \gg 1$. By multiplying the amplitude by time, we can see that the oscillation is more or less $1/t$, consistent with the analytic approximate solution.

```
Plot[Evaluate[t ϕ[t] /. sol[[1]]], {t, 10000, 100000}]
```



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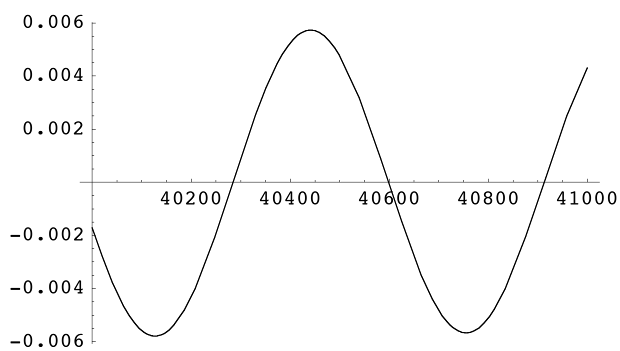
To compare the numerical and anlaytic solutions head-to-head, we need to do some more work. We fix the parameters by looking at one period in the numerical solution

```
Plot[Evaluate[ϕ[t] /. sol[[1]]], {t, 40000, 41000}]
```



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t_0 = t /. FindRoot[ϕ[t] == 0 /. sol[[1]], {t, 40300}][[1]]

40284.3

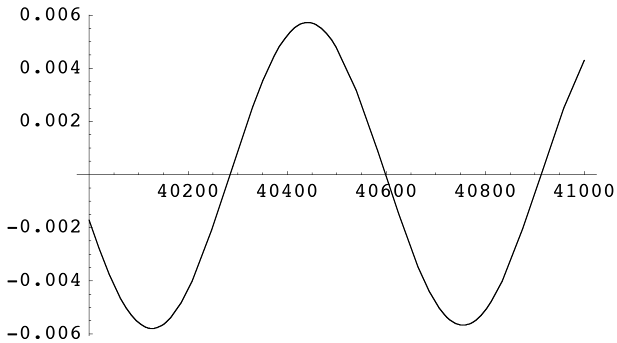
Evaluate$\left[\phi[t] \text{ /. sol[[1]] /. } \left\{t \to t_0 + \frac{\pi}{2}\frac{1}{m}\right\} \text{ /. } \{m \to 10^{-2}\}\right]$

0.00573029

Then the approximate analytic

```
Plot[t1/(t + t1 - t0 - (1/m)(π/2)) φ1 Cos[m (t - t0) - π/2] /. {t1 → 2 Sqrt[2/3] MPl/(m φ1)} /. {m → 10^-2, MPl → 1} /.
  {φ1 -> 0.005730285458006179`, t0 -> 40284.31696953443`}, {t, 40000, 41000}]
```

$$\text{Plot}\left[\frac{t_1}{t + t_1 - t_0 - \frac{1}{m}\,\frac{\pi}{2}}\,\phi_1 \,\text{Cos}\left[m\,(t - t_0) - \frac{\pi}{2}\right] \,/.\, \left\{t_1 \to 2\sqrt{\frac{2}{3}}\,\frac{\text{MPl}}{m\,\phi_1}\right\} \,/.\, \{m \to 10^{-2}, \text{MPl} \to 1\} \,/.\, \{\phi_1 \text{->} 0.005730285458006179\text{`}, t_0 \text{->} 40284.31696953443\text{`}\}, \{t, 40000, 41000\}\right]$$



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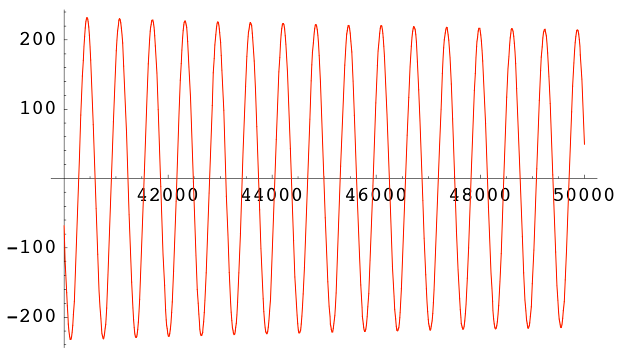
```
Plot[Evaluate[t φ[t] /. sol[[1]]], {t, 40000, 50000}, PlotStyle → RGBColor[1, 0, 0]]
```



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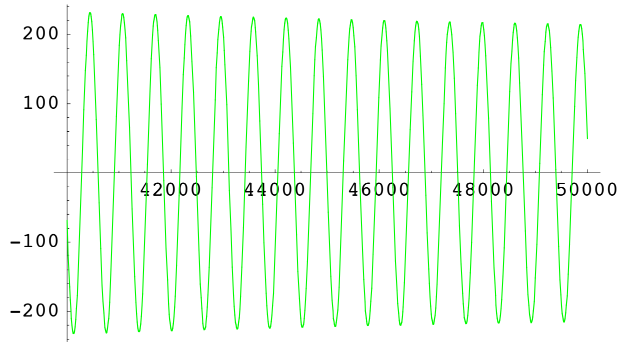
$$\texttt{Plot}\left[t\,\frac{t_1}{t+t_1-t_0-\frac{1}{m}\,\frac{\pi}{2}}\,\phi_1\,\texttt{Cos}\left[m\,(t-t_0)-\frac{\pi}{2}\right]\;/.\;\left\{t_1\to 2\,\sqrt{\frac{2}{3}}\,\frac{\texttt{MPl}}{m\,\phi_1}\right\}\;/.\;\{m\to 10^{-2},\,\texttt{MPl}\to 1\}\;/.\;\{\phi_1\;\texttt{->}\;0.005730285458006179\text{`},\;t_0\;\texttt{->}\;40284.31696953443\text{`}\},\;\{t,\,40000,\,50000\},\;\texttt{PlotStyle}\to\texttt{RGBColor}[0,1,0]\right]$$



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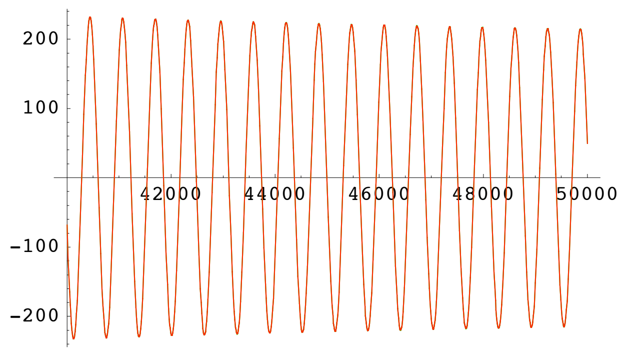

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They agree very well with each other.

Optional

It is too cumbersome to write many equations in *Mathematica*. It will be provided as a separate PDF file.