HW #6

1. Inflaton

(a) slow-roll regime

In the slow-roll regime, we neglect the kinetic energy as well as $\dot{\phi}$ term in the equation of motion. Then

$$H^2 = \frac{8\pi}{3} G_N \frac{m^2}{2} \phi^2, \quad 3 H \phi + m^2 \phi = 0.$$  

We use $8 \pi G_N = M_{Pl}^{-2}$. Putting them together,

$$3 \frac{m}{\sqrt{6} M_{Pl}} \frac{\dot{\phi}^2}{\phi} + m^2 \phi = 0,$$

and hence

$$d \phi = -\sqrt{\frac{2}{3}} m M_{Pl} d t.$$  

The solution is simply

$$\phi(t) = \phi(0) - \sqrt{\frac{2}{3}} m M_{Pl} t.$$  

Using this solution, the kinetic energy is

$$\frac{1}{2} \dot{\phi}^2 = \frac{1}{3} m^2 M_{Pl}^2,$$

while the potential energy is

$$m^2 \phi^2.$$  

Therefore, if $\phi \gg M_{Pl}$, we see that the kinetic energy is indeed smaller than the potential energy. At the same time, $\phi = 0$ for the above solution and indeed is negligible compared to other non-vanishing terms. The scale factor can be obtained from the Friedmann equation

$$H^2 = \frac{R}{R^*} = \frac{1}{3 M_{Pl}^2} \frac{1}{2} m^2 \phi^2,$$

$$\frac{R}{R^*} = \frac{1}{\sqrt{6} M_{Pl}} \ m \ \phi,$$

$$d \log R = \frac{1}{\sqrt{6} M_{Pl}} \ m \ \phi \ dt$$

and hence

$$\log \frac{R(t)}{R(0)} = \frac{1}{\sqrt{6} M_{Pl}} \ m (\phi(0) t - \frac{1}{\sqrt{6} M_{Pl}} t^2)$$

When $\phi \gg M_{Pl}$, the first term in the parentheses dominates and the scale factor grows exponentially with time.
(b) oscillating regime

In the oscillating regime $\phi \ll M_{Pl}$ and $m \gg 1$, we study the equation of motion

$$\ddot{\phi} + 3 H \dot{\phi} + m^2 \phi = 0,$$

$$H^2 = \frac{1}{3 M_{Pl}^2} \left( \frac{1}{2} \dot{\phi}^2 + m^2 \phi^2 \right)$$

with the ansatz

$$\phi(t) = \phi(t_1) \frac{\nu}{t} \cos(m(t-t_1)).$$

$$\dot{\phi} = \phi(t_1) \frac{\nu}{t} \left( -m \sin(m(t-t_1)) + 2 \frac{m}{\nu} \sin(m(t-t_1)) + \frac{2}{\nu^2} \right)$$

The leading term in $1/(m t)$ is the first term in the parantheses which is canceled by $m^2 \phi$ term. To see if the next-leading term is also cancelled in the equation of motion, we work out

$$\phi = \phi(t_1) \frac{\nu}{t} \left( -m \sin(m(t-t_1)) - \frac{1}{n} \cos(m(t-t_1)) \right),$$

$$H^2 = \frac{1}{3 M_{Pl}^2} \left( \frac{1}{2} \phi(t_1)^2 \frac{\nu}{\nu^2} \left( m^2 + 2 m \frac{m}{\nu} \sin(m(t-t_1)) \cos(m(t-t_1)) + \frac{1}{\nu^2} \cos^2 m(t-t_1) \right) \phi(t_1)^2 \frac{\nu}{m} \left( -m \sin(m(t-t_1)) - \frac{1}{n} \cos(m(t-t_1)) \right) = 0.$$

The next leading terms are

$$\phi(t_1) \frac{\nu}{t} \left( 2 \frac{m}{\nu} \sin(m(t-t_1)) \right) - 3 \frac{1}{\sqrt{3} M_{Pl}^2} \phi(t) \frac{\nu}{t} m \phi(t_1)^2 \frac{\nu}{m} m \sin(m(t-t_1)) = 0.$$

This is satisfied if

$$2 - 3 \frac{1}{\sqrt{3} M_{Pl}^2} \nu \phi(t_1) = 0.$$

$$t_1 = 2 \sqrt{\frac{2}{3} \frac{M_{Pl}}{m \phi(t_1)}}.$$

With this choice, the equation of motion is satisfied both for the leading and the next-leading terms in the expansion in $1/m t$.

The expansion rate to the leading order is

$$H^2 = \frac{1}{3 M_{Pl}^2} \left( \frac{1}{2} \dot{\phi}^2 \frac{\nu}{\nu^2} m^2 \right) = \frac{4}{2} \frac{M_{Pl}^2}{\nu} \frac{1}{\nu^2} m^2 = \frac{4}{9} \frac{1}{\nu^2}.$$

This differential equation can be integrated easily and we find

$$R = R(t_1) \left( \frac{t_1}{t} \right)^{2/3}.$$

Therefore the energy density scales as

$$\rho = 3 M_{Pl}^2 H^2 = 3 M_{Pl}^2 \frac{4}{9} \frac{1}{\nu^2} = \frac{4}{3} \frac{1}{\nu^2} \left( \frac{R(t_1)}{R} \right)^3.$$

Indeed, the energy density is that of matter-dominated universe.

(c) numerical solution

We solve the differential equation numerically. A word of caution is that this is actually a not very safe thing to do if the solution oscillates crazy like this one. The numerical errors may build up. Mathematica seems to handle it fairly well, though.

I didn't specify the boundary conditions. Normally, we take the initial time derivative to vanish, but of course we don't really know what the right boundary conditions are.
\[
sol = \text{NDSolve}\left[\left\{ f''[t] + 3H f'[t] + m^2 f[t] = 0 \right\}, \left\{ H = \sqrt{\frac{1}{3 M_{\text{Pl}}^2} \frac{1}{2} (f'[t]^2 + m^2 f[t]^2)} \right\} \right]/.
\left\{ M_{\text{Pl}} \rightarrow 1, m \rightarrow 10^{-2}, f[0] = 100, f'[0] = 0 \right\}, \{ t, 0, 100000 \} \right]
\]

\[
\left[ \{ f \rightarrow \text{InterpolatingFunction}[\{ \{ 0., 100000. \} \}, <>] \} \right]
\]

\[
\text{Plot}\left[ \text{Evaluate}[f[t] /. \text{sol}[1]], \{ t, 0, 20000 \}, \text{PlotRange} \rightarrow \{-1, 100\}, \text{PlotStyle} \rightarrow \text{RGBColor}[0, 1, 0] \right]
\]

- Graphics -

The slow-roll solution does not have vanishing time derivative at the initial time.

\[
\text{Plot}\left[ \sqrt{\frac{2}{3} m M_{\text{Pl}} t} \right. \left./. \left\{ \phi_0 \rightarrow 100, m \rightarrow 10^{-2}, M_{\text{Pl}} \rightarrow 1 \right\}, \{ t, 0, 13000 \}, \text{PlotRange} \rightarrow \{-1, 100\}, \text{PlotStyle} \rightarrow \text{RGBColor}[1, 0, 0] \right]
\]

- Graphics -
The fact that they agree so well with each other is a demonstration that the inflation is quite insensitive to the initial values; the solution quickly approaches the slow-roll solution.

Note that the time $t$ in this solution is not the same as the time $t$ in the analytic solution in the oscillating regime; they are offset by the contribution from the slow-roll regime. But the offset is quickly forgotten as time goes on $mt \gg 1$. By
Note that the time \( t \) in this solution is not the same as the time \( t \) in the analytic solution in the oscillating regime; they are offset by the contribution from the slow-roll regime. But the offset is quickly forgotten as time goes on.

By multiplying the amplitude by time, we can see that the oscillation is more or less \( 1/t \), consistent with the analytic approximate solution.

\[
\text{Plot}\left[\text{Evaluate}\left[t \phi(t) \cdot \text{sol}[[1]]\right], \{t, 10000, 100000\}\right]
\]

To compare the numerical and analytic solutions head-to-head, we need to do some more work. We fix the parameters by looking at one period in the numerical solution

\[
\text{Plot}\left[\text{Evaluate}\left[\phi(t) \cdot \text{sol}[[1]]\right], \{t, 40000, 41000\}\right]
\]

\[
\text{t}_0 = t \cdot \text{FindRoot}\left[\phi(t) = 0 \cdot \text{sol}[[1]], \{t, 40300\}\right][[1]]
\]

\[40284.3\]

\[
\text{Evaluate}\left[\phi(t) \cdot \text{sol}[[1]] / \left\{ t \rightarrow t_0 + \frac{\pi}{2} \cdot \frac{1}{m} \right\} \cdot \{m \rightarrow 10^{-2}\}\right]
\]

\[0.00573029\]

Then the approximate analytic
Plot\[
\text{Plot}\left[\frac{t_1}{t + t_1 - t_0 - \frac{1}{m} \frac{\pi}{2}} \cdot \phi_1 \cos\left(m (t - t_0) - \frac{\pi}{2}\right) \cdot \left\{t_1 \rightarrow 2 \sqrt{\frac{2}{3} \frac{\text{MPl}}{m \phi_1}}\right\} / \cdot \left\{m \rightarrow 10^{-2}, \text{MPl} \rightarrow 1\right\} / \cdot \phi_1 \rightarrow 0.005730285458006179^\circ, t_0 \rightarrow 40284.31696953443^\circ\right\}, \{t, 40000, 41000\}\]

- Graphics -

Plot[Evaluate[t \phi[t] /. sol[[1]]], \{t, 40000, 50000\}, PlotStyle \rightarrow \text{RGBColor}[1, 0, 0]]

- Graphics -
They agree very well with each other.

Optional

It is too cumbersome to write many equations in Mathematica. It will be provided as a separate PDF file.