# HW #2

## 1. Luminosity-Magnitude Relation

### (a)

The Friedmann equation says $\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G_N(\rho_M + \rho_\Lambda) - \frac{k}{R^2}$. Using the current values $\frac{8\pi}{3} G_N \rho_M = H_0{}^2 \Omega_M$, $\frac{8\pi}{3} G_N \rho_\Lambda = H_0{}^2 \Omega_\Lambda$, $\frac{-k}{R^2} = H_0{}^2 \Omega_k$, we can rewrite it as $\left(\frac{\dot{R}}{R}\right)^2 = H_0{}^2\left(\Omega_M (\frac{R_0}{R})^3 + \Omega_\Lambda + \Omega_k (\frac{R_0}{R})^2\right)$. Therefore, $\frac{dR}{R}\left(\Omega_M (\frac{R_0}{R})^3 + \Omega_\Lambda + \Omega_k (\frac{R_0}{R})^2\right)^{-1/2} = H_0\, dt$. Because $\frac{R_0}{R} = 1 + z$, we find
$\frac{dt}{R} = \frac{1}{H_0} \frac{dR}{R^2} (\Omega_M (1+z)^3 + \Omega_\Lambda + \Omega_k (1+z)^2)^{-1/2} = \frac{1}{H_0 R_0} dz\, (\Omega_M (1+z)^3 + \Omega_\Lambda + \Omega_k (1+z)^2)^{-1/2}$
Namely,
$H_0 R_0 \int_{t_1}^{t_0} \frac{dt}{R(t)} = \int_0^z \frac{dz'}{(\Omega_M (1+z')^3 + \Omega_\Lambda + \Omega_k (1+z')^2)^{1/2}}$ .

### (b)

We first verify the small $z$ behavior. Using $\Omega_M + \Omega_\Lambda + \Omega_k = 1$,
$(\Omega_M (1+z')^3 + \Omega_\Lambda + \Omega_k (1+z')^2)^{-1/2} = (1 + 3\,\Omega_M z' + 2\,\Omega_k z' + O(z')^2)^{-1/2}$
$= 1 - (\frac{3}{2} \Omega_M + \Omega_k) z' = 1 - (1 + \frac{1}{2} \Omega_M - \Omega_\Lambda) z'$
and hence
$H_0 R_0(\arcsin r,\ r,\ \operatorname{arcsinh} r) = z - \frac{1}{2} (1 + \frac{1}{2} \Omega_M - \Omega_\Lambda) z^2 + O(z^3)$
We also expand the l.h.s. in $r$ up to $O(r^2)$, and to this order, they are all just $H_0 R_0 r + O(r^2)$. We rewrite it in terms of the luminosity distance defined by $d_L = R_0 r(1+z)$,
$H_0 d_L = (z - \frac{1}{2} (1 + \frac{1}{2} \Omega_M - \Omega_\Lambda))(1+z) = z + \frac{1}{2} (1 - \frac{1}{2} \Omega_M + \Omega_\Lambda) z^2 + O(z^3)$
Compared to the definition of the decceleration parameter
$H_0 d_L = z + \frac{1}{2} (1 - q_0) z^2 + O(z^3)$, we find
$q_0 = \frac{1}{2} \Omega_M - \Omega_\Lambda$

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## (1) sCDM

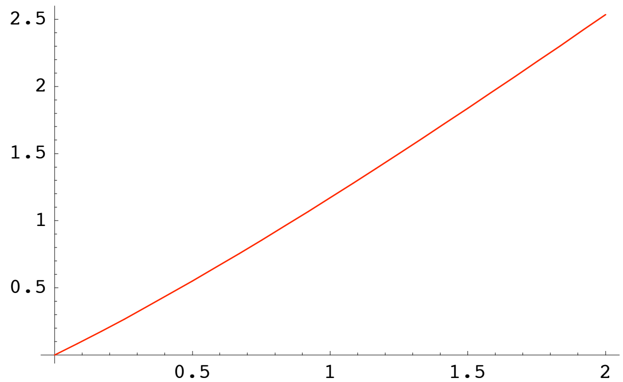
We take $\Omega_M = 1$, $\Omega_\Lambda = \Omega_k = 0$.

First, exact numerical integration:

```
In[22]:= Plot[
```

$$(1+z)\ \mathrm{NIntegrate}\left[\frac{1}{\sqrt{\Omega_M\,(1+y)^3+\Omega_\Lambda+(1-\Omega_M-\Omega_\Lambda)\,(1+y)^2}}\ /.\ \{\Omega_M\to 1,\ \Omega_\Lambda\to 0\},\ \{y,0,z\}\right],$$

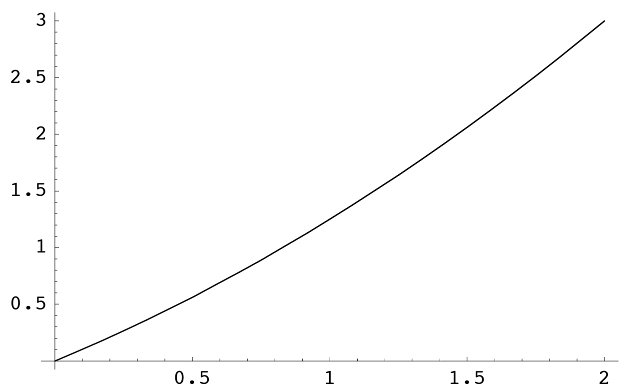
```
{z, 0, 2}, PlotStyle → RGBColor[1, 0, 0]]
```



```
Out[22]= - Graphics -
```

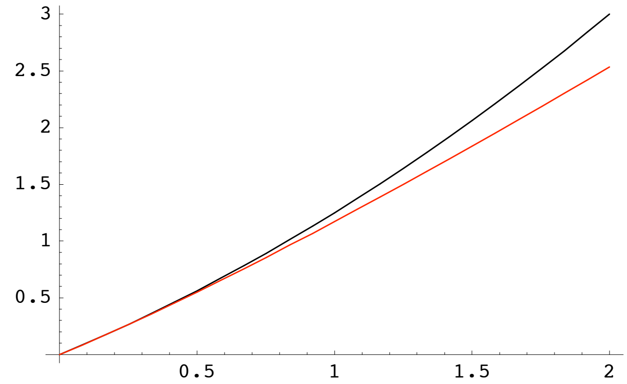
Approximation up to $z^2$:

$$\mathrm{In[23]:=}\ \ \mathrm{Plot}\left[z+\frac{1}{2}\left(1-\left(\frac{1}{2}\,\Omega_M-\Omega_\Lambda\right)\right)z^2\ /.\ \{\Omega_M\to 1,\ \Omega_\Lambda\to 0\},\ \{z,0,2\}\right]$$



```
Out[23]= - Graphics -
```

```
In[24]:=  Show[%, %%]
```




```
Out[24]=  - Graphics -
```

## (2) oCDM

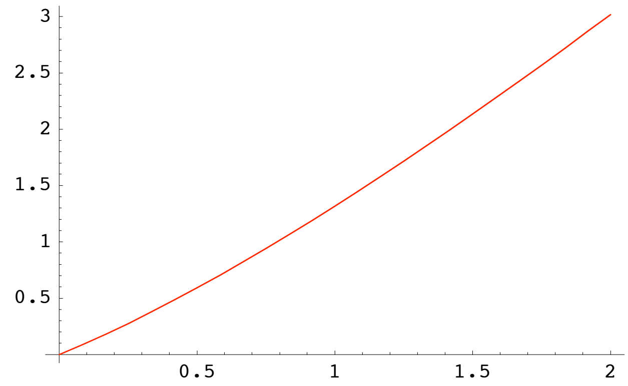
We take $\Omega_M = 0.25$, $\Omega_\Lambda = 0$, $\Omega_k = 0.75$

First, exact numerical integration:

In[25]:= **Plot**[

$$(1+z)\ \mathbf{NIntegrate}\left[\frac{1}{\sqrt{\Omega_M\,(1+y)^3+\Omega_\Lambda+(1-\Omega_M-\Omega_\Lambda)\,(1+y)^2}}\ /.\ \{\Omega_M\to 0.25,\ \Omega_\Lambda\to 0\},\ \{y,\ 0,\ z\}\right],$$

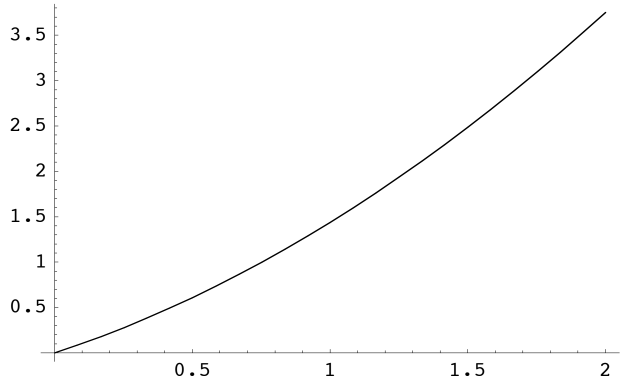
**{z, 0, 2}, PlotStyle → RGBColor[1, 0, 0]**]




```
Out[25]=  - Graphics -
```

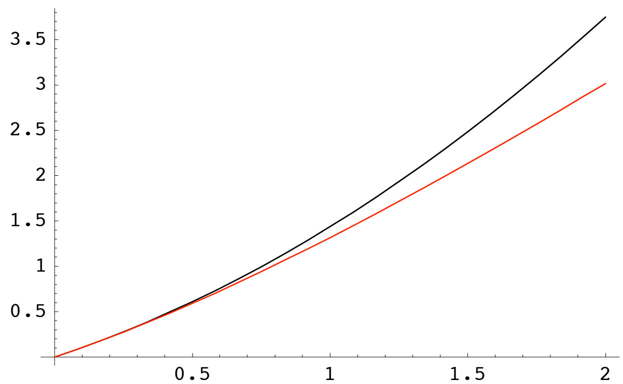
Approximation up to $z^2$:

In[26]:= `Plot[z + 1/2 (1 - (1/2 Ω_M - Ω_Λ)) z^2 /. {Ω_M → 0.25, Ω_Λ → 0}, {z, 0, 2}]`




Out[26]= - Graphics -

In[27]:= `Show[%, %%]`



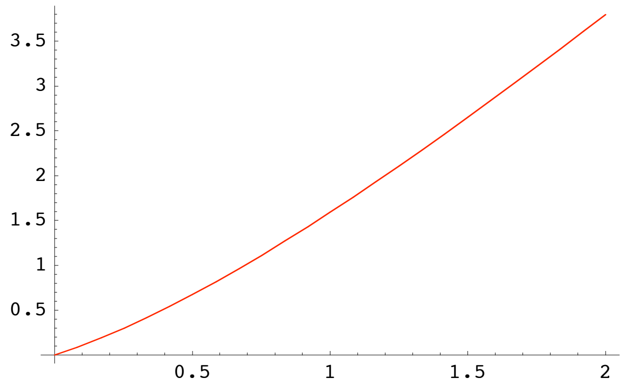

Out[27]= - Graphics -

## (3) ΛCDM

We take $\Omega_M = 0.25$, $\Omega_\Lambda = 0.75$, $\Omega_k = 0$.

First, exact numerical integration:

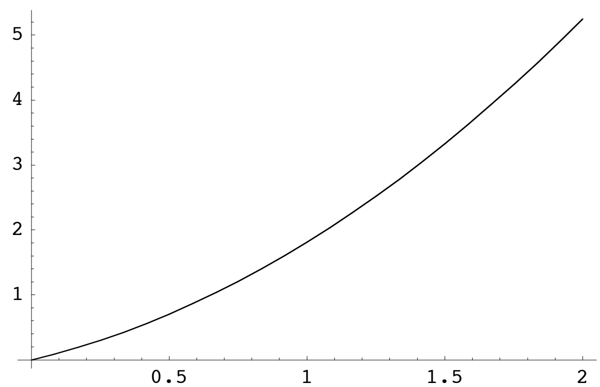
```
In[33]:= Plot[(1 + z)
```
$$\text{NIntegrate}\left[\frac{1}{\sqrt{\Omega_M\,(1+y)^3+\Omega_\Lambda+(1-\Omega_M-\Omega_\Lambda)\,(1+y)^2}}\;/.\;\{\Omega_M\to 0.25,\ \Omega_\Lambda\to 0.75\},\ \{y,0,z\}\right],$$
```
  {z, 0, 2}, PlotStyle → RGBColor[1, 0, 0]]
```




```
Out[33]= - Graphics -
```

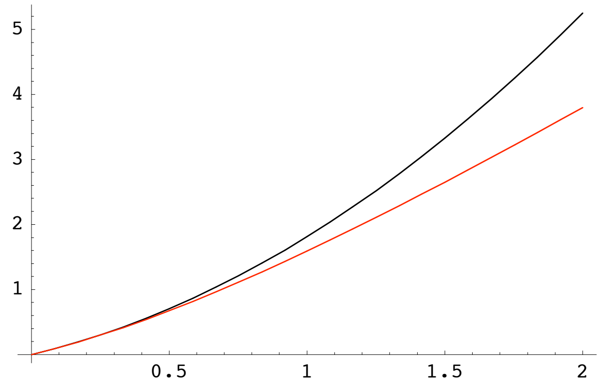
Approximation up to $z^2$:

$$\text{In[34]:= Plot}\left[z+\frac{1}{2}\left(1-\left(\frac{1}{2}\,\Omega_M-\Omega_\Lambda\right)\right)z^2\;/.\;\{\Omega_M\to 0.25,\ \Omega_\Lambda\to 0.75\},\ \{z,0,2\}\right]$$




```
Out[34]= - Graphics -
```

```
In[35]:= Show[%, %%]
```




```
Out[35]= - Graphics -
```

## (3)

The apparent brightness goes as ${d_L}^{-2}$, while the magnitude is $2.5 \log_{10} {d_L}^2$, and hence goes as $5 \log_{10} d_L$. We do not know $H_0$ or absolute brightness of the Type-IA supernovae very well, but we do not need to. We just scale $5 \log_{10} d_L$ up and down to match the data of the nearby supernovae, and see which curve fits the high-$z$ supernovae. This is the nice trick that allowed Saul Perlmutter and company to discover the accelerating expansion.

I used a different plotting program to plot them.