HW #7, due Mar 10

1. Sigma Model The sigma model describes the spontaneous breaking of chiral $SU(2)_L \times SU(2)_R$ symmetry. The field Φ is a two-by-two matrix

$$\Phi = \sigma + i\vec{\pi}\cdot\vec{\tau} = \begin{pmatrix} \sigma + i\pi^3 & i\pi^1 + \pi^2 \\ i\pi^1 - \pi^2 & \sigma - i\pi^3 \end{pmatrix} = \begin{pmatrix} \sigma + i\pi^0 & i\sqrt{2}\,\pi^+ \\ i\sqrt{2}\,\pi^- & \sigma - i\pi^0 \end{pmatrix}, \qquad (1)$$

where σ, π^i $(i = 1, 2, 3)$ are real Klein–Gordon fields. The $SU(2)_L \times SU(2)_R$ transformation is given by

$$\Phi \to \Phi' = V_L \Phi V_R^\dagger, \qquad (2)$$

where V_L, V_R are unitary matrices with unit determinant. Answer the following questions.

(a) Verify that $\mathrm{Tr}\Phi^\dagger\Phi = 2(\sigma^2 + \vec{\pi}^2)$ is invariant under $SU(2)_L \times SU(2)_R$.

(b) Show that the kinetic terms for σ, $\vec{\pi}$ can be conveniently written as $\frac{1}{4}\mathrm{Tr}\partial_\mu\Phi^\dagger\partial^\mu\Phi$. Verify that it is $SU(2)_L \times SU(2)_R$ invariant.

(c) Minimize the potential $V = -\mu^2(\sigma^2 + \vec{\pi}^2)^2 + \lambda(\sigma^2 + \vec{\pi}^2)^4$ when $\sigma_0 = \langle\sigma\rangle > 0$ and $\vec{\pi} = 0$. (Even if $\vec{\pi} \neq 0$, one can always find an $SU(2)_L \times SU(2)_R$ transformation to make it vanish.)

(d) Expand Φ around its minimum in V, *i.e.*, $\Phi = \sigma_0 + \sigma' + i\vec{\pi}\cdot\vec{\tau}$, and calculate the mass of σ' and $\vec{\pi}$. (Identify the quadratic terms and equate them with $m_{\sigma'}^2\sigma'^2/2$ etc.)

(e) Now we introduce finite masses of up and down quarks. Using the matrix

$$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}, \qquad (3)$$

we add $\Delta V = -\rho^2\mathrm{Tr}(M^\dagger\Phi + h.c.)$. Minimize the potential and calculate the masses of σ' and $\vec{\pi}$ (you don't need to keep terms of higher orders in M.) Show that the pion mass *squared* is proportional to the quark mass.

Note The σ' particle does not exist in nature (at best controversial). It is a mere convenience in writing down this toy model. To describe the dynamics of pions, we would like to eliminate σ' from the theory. We can do so by considering the limit $\lambda \to \infty$ with keeping σ_0 finite, which makes σ' infinitely heavy. Then the potential restricts $\sigma^2 + \vec{\pi}^2 = \sigma_0^2$, which in turn let us solve for σ in terms of $\vec{\pi}$. In this limit, one can use an alternative parametrization of $\Phi \equiv \sigma_0 U$, where

$$U = e^{i\vec{\pi}'\cdot\vec{\tau}/\sigma_0} = \cos\frac{|\vec{\pi}'|}{\sigma_0} + i\frac{\vec{\pi}'\cdot\vec{\tau}}{|\vec{\pi}'|}\sin\frac{|\vec{\pi}'|}{\sigma_0} \tag{4}$$

which is related to the original one by

$$\frac{\vec{\pi}}{\sigma_0} = \frac{\vec{\pi}'}{|\vec{\pi}'|}\sin\frac{|\vec{\pi}'|}{\sigma_0} = \frac{\vec{\pi}'}{\sigma_0} + O\left(\frac{\vec{\pi}'}{\sigma_0}\right)^3. \tag{5}$$

Therefore one can use either of the parametrizations to describe pions with identical physics. Drop primes hereafter.

(f) Write down the kinetic term and the mass term in terms of the matrix field $U = e^{i\vec{\pi}'\cdot\vec{\tau}/\sigma_0}$. This is the chiral Lagrangian. (It is non-renormalizable, but is good enough to describe physics at energies less than $\sigma_0 = F_\pi$ in power series expansion of p^μ/F_π.)

(g) Using the field U, it is easy to generalize it to the case of $SU(3)_L \times SU(3)_R$. The field is parameterized as $U = e^{i2\Pi/F_\pi}$, where

$$\Pi = \pi^a T^a = \frac{1}{2}\begin{pmatrix} \pi^0 + \eta/\sqrt{3} & \sqrt{2}\,\pi^+ & \sqrt{2}\,K^+ \\ \sqrt{2}\,\pi^- & -\pi^0 + \eta/\sqrt{3} & \sqrt{2}\,K^0 \\ \sqrt{2}\,K^- & \sqrt{2}\,\overline{K}^0 & -2\eta/\sqrt{3} \end{pmatrix}. \tag{6}$$

Using the mass matrix

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}, \tag{7}$$

calculate the mass spectrum of pseudo-scalar mesons.

(h) Add an electromagnetic self-energy Δ to π^+ and K^+ mass-squared. Determine $\rho^2 F_\pi m_u$, $\rho^2 F_\pi m_d$, $\rho^2 F_\pi m_s$, Δ from $m_{\pi^+} = 139.6$, $m_{\pi^0} = 135.0$, $m_{K^0} = 497.7$, $m_{K^+} = 493.7$ MeV. Use them as inputs to calculate η mass and compare it to the data $m_\eta = 547.3$ MeV.