# HW #4, due Feb 18

**1.** The cross section of the process $e^+e^- \to q\bar{q}g$ is given by

$$\sigma_{3jets} = \sigma_{pt}\left(N_c \sum_f Q_f^2\right) C_F \frac{\alpha_s}{2\pi} \int dx_1 dx_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}, \tag{1}$$

where $\sigma_{pt} = \sigma(e^+e^- \to \mu^+\mu^-) = 4\pi^2\alpha^2/3s$. The quadratic Casimir is given by $C_F = T^aT^a = \frac{N_c^2-1}{2N_c}$ for $SU(N_c)$ groups and and is 4/3 for $N_c = 3$.* The kinematical variables are defined by $x_1 = E_q/E_{\text{beam}}$ and $x_2 = E_{\bar{q}}/E_{\text{beam}}$. Another useful variable is $x_g = E_g/E_{\text{beam}}$ with the constraint $x_1 + x_2 + x_g = 2$ due to the energy conservation. In JADE jet clustering algorithm, the integration region is bounded by $x_{1,2,g} < 1 - y_{cut}$. The bound on $x_g$ can be re-expressed as $x_1 + x_2 > 1 + y_{cut}$. On the other hand, the total cross section up to $O(\alpha_s)$ is given by

$$\sigma_{tot} = \sigma_{pt}\left(N_c \sum_f Q_f^2\right)\left(1 + \frac{\alpha_s}{\pi}\right). \tag{2}$$

Quarks and gluons are all regarded massless.

(a) Show that $m_{qg}^2 = s(1-x_2)$ (just kinematics). This is why the JADE jet clustering algotrithm, which requires $m_{qg}^2 > y_{cut}s$ to have two jets resolved, gives the bound $x_2 < 1 - y_{cut}$. Since it is hard to distinguish jets from $q$, $\bar{q}$ $g$ partons experimentally, similar bounds should be imposed on $x_1$, $x_g$ as well.

(b) Integrate (either analytically or numerically; use Mathematica, Maple, Matlab, or whatever you want) the cross section and plot the three-jet fraction $R_3(y_{cut}) = \sigma_{3jets}/\sigma_{tot}$ and the two-jet fraction $R_2 = 1 - R_3$ as functions of $y_{cut}$ on the log (linear) scale for the horizontal (vertical) axis, for $y_{cut} > 10^{-2}$. (Below this value, the four-jet fraction becomes appreciable, but we can't study this unless we have expressions up to $O(\alpha_s)^2$.)

(c) Compare the result to Figure 1 in `http://arXiv.org/abs/hep-ex/0001055`. Read off rough numbers for $\alpha_s$ for $\sqrt{s} = 35, 91, 189$ GeV.

Note At $\sqrt{s} = 91$ GeV, the value of $\alpha_s$ is somewhat larger than what was obtained from $\sigma_{tot}$, $\alpha_s(m_Z) = 0.119 \pm 0.003$. This is because the integral Eq. (1) is dominated when $x_{1,2} \sim 1 - y_{cut}$ and the smallest invariant mass among three possible combinations of two of the jets is roughly $m_{ij}^2 \sim y_{cut}s$, which should be regarded as the typical energy scale of the problem. Therefore, the extracted value of $\alpha_s(\mu)$ is given roughly at $\mu^2 \sim y_{cut}m_Z^2 < m_Z^2$. But this intuitive argument can be verified only by going to the NLO (Next-to-the-Leading-Order) calculation at $O(\alpha_s^2)$.

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*This is what was called $C_2$ in HW #3. There, I wanted to emphasize the "quadratic" nature but in the QCD community it is normally called $C_F$.