HW #2, due Feb 4

1. We woud like to check the gauge invariance of the QED Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi, \tag{1}$$

where $D_\mu = \partial_\mu - ieQA_\mu$. The gauge transformation is given by

$$\psi'(x) = e^{iQ\omega(x)}\psi(x), \qquad A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\omega \tag{2}$$

(a) Show that $D'_\mu\psi' = e^{iQ\omega}D_\mu\psi$.

(b) Show that $[D_\mu, D_\nu]\psi = -ieQF_{\mu\nu}\psi$. Note that this implies that $[D'_\mu, D'_\nu]\psi' = -ieQF'_{\mu\nu}\psi' = e^{iQ\omega}(-ieQF_{\mu\nu}\psi)$ and hence $F'_{\mu\nu} = F_{\mu\nu}$.

(c) Using the above results, show that $\mathcal{L}(\bar{\psi}', \psi', A') = \mathcal{L}(\bar{\psi}, \psi, A)$.

2. We woud like to check the gauge invariance of the Lagrangian of non-abelian gauge theories

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi, \tag{3}$$

where $D_\mu = \partial_\mu - igA_\mu$ with the matrix form $A_\mu = A^a_\mu T^a$. The gauge transformation is given by

$$\psi'(x) = U(x)\psi(x), \qquad A'_\mu(x) = U(x)A_\mu(x)U(x)^{-1} + \frac{i}{g}U\partial_\mu U^{-1}. \tag{4}$$

(a) Show that $D'_\mu\psi' = UD_\mu\psi$.

(b) Show that $D'_\mu = UD_\mu U^{-1}$.

(c) Define $F_{\mu\nu}$ by $[D_\mu, D_\nu]\psi = -igF_{\mu\nu}\psi$. Show that $[D'_\mu, D'_\nu]\psi' = -igF'_{\mu\nu}\psi' = U(-igF_{\mu\nu}\psi)$ and hence $F'_{\mu\nu} = UF_{\mu\nu}U^{-1}$.

(d) Using the above results, show that $\mathcal{L}(\bar{\psi}', \psi', A') = \mathcal{L}(\bar{\psi}, \psi, A)$.

(e) Show that $F_{\mu\nu} = F^a_{\mu\nu}T^a = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu)T^a$.

(f) Under infinitesimal transformations $U = e^{i\omega^a T^a} = 1 + i\omega^a T^a + O(\omega^2)$, show that $A'_\mu = A_\mu + \frac{1}{g}D_\mu\omega + O(\omega)^2$, where $\omega = \omega^a T^a$ and $D_\mu\omega = \partial_\mu\omega - ig[A_\mu, \omega]$. Similarly, show that $F'_{\mu\nu} = F_{\mu\nu} - i[F_{\mu\nu}, \omega] + O(\omega)^2$.