Explicit Calculations of the Vertex Funcion

We use the same notation as in the textbook page 189. The incoming (outgoing) electron has four-momentum p (p'), and the Coulomb field kicks the electron with a momentum transfer $q = p' - p$. The vertex function does not include $-ie$ factor in its definition, so that its lowest-order piece is γ^μ. Writing the vertex function as $\Gamma^\mu(p', p) = \gamma^\mu + \delta\Gamma^\mu(p', p)$,

$$\begin{aligned}\delta\Gamma^\mu(p', p) &= \int \frac{d^4k}{(2\pi)^4}(-ie\gamma_\nu)\frac{-i}{(p-k)^2 - \mu^2}\frac{i}{\not{k}' - m}\gamma^\mu \frac{i}{\not{k} - m}(-ie\gamma^\nu) \\ &= -ie^2 \int \frac{d^4k}{(2\pi)^4}\frac{1}{(p-k)^2 - \mu^2}\frac{\gamma_\nu(\not{k}' + m)\gamma^\mu(\not{k} + m)\gamma^\nu}{(k'^2 - m^2)(k^2 - m^2)} \end{aligned} \tag{1}$$

By using the identities $\gamma_\nu \not{a}\gamma^\nu = -2\not{a}$, $\gamma_\nu \not{a}\not{b}\gamma^\nu = 4(a\cdot b)$, and $\gamma_\nu \not{a}\not{b}\not{c}\gamma^\nu = -2\not{c}\not{b}\not{a}$, the numerator can be simplified to

$$\gamma_\nu(\not{k}' + m)\gamma^\mu(\not{k} + m)\gamma^\nu = -2\not{k}\gamma^\mu\not{k}' + 4m(k + k')^\mu - 2m^2\gamma^\mu \tag{2}$$

On the other hand, three propagators can be combined to a single one using the Feynman parameters

$$\frac{1}{ABC} = 2\int_0^1 d^3z\delta(\sum_i^3 z_i - 1)\frac{1}{(z_1A + z_2B + z_3C)^3} \tag{3}$$

Let us work out the combined propagator first. Using $z_1 + z_2 + z_3 = 1$, $p^2 = m^2$, $2p\cdot q = (p+q)^2 - p^2 - q^2 = -q^2$ and $k' = k + q$,

$$\begin{aligned}&z_1((p-k)^2 - \mu^2) + z_2(k'^2 - m^2) + z_3(k^2 - m^2) \\ &= k^2 - 2z_1k\cdot p + z_1m^2 - z_1\mu^2 + 2z_2k\cdot q + z_2q^2 - z_2m^2 - z_3m^2 \\ &= (k - z_1p + z_2q)^2 - z_1^2m^2 + 2z_1z_2p\cdot q - z_2^2q^2 + z_1m^2 - z_1\mu^2 + z_2q^2 - (z_2 + z_3)m^2 \\ &= l^2 - (1 - z_1)^2m^2 - z_1\mu^2 + z_2z_3q^2 \end{aligned} \tag{4}$$

where I introduced

$$l = k - z_1p + z_2q. \tag{5}$$

By using l and Feynman parameters, we obtain

$$\delta\Gamma^\mu(p', p) = -ie^2 2\int_0^1 d^3z\delta(\sum_i^3 z_i - 1)\int \frac{d^4l}{(2\pi)^4}\frac{-2\not{k}\gamma^\mu\not{k}' + 4m(k + k')^\mu - 2m^2\gamma^\mu}{[l^2 - (1 - z_1)^2m^2 - z_1\mu^2 + z_2z_3q^2]^3}. \tag{6}$$

Let us work on the numerator. The aim is to rewrite it as a linear combination of γ^μ piece, $i\sigma^{\mu\nu}q_\nu$ piece and q^μ piece. The reason is the following. It is only γ^μ piece which receives the wave function renormalization at this order, as we will see later. Therefore, $i\sigma^{\mu\nu}q_\nu$ piece is finite by itself. Furthermore, this is the piece which generates the anomalous magnetic moment, and hence it is useful to separate them from the physics point of view as well. We will benefit from the fact that the $\delta\Gamma^\mu$ is sandwiched between $\bar{u}(p')$ and $u(p)$.

First the $-2\not{k}\gamma^\mu\not{k}'$ term. By going to l variable,

$$\begin{aligned}
&-2\not{k}\gamma^\mu\not{k}' \\
&= -2(\not{l}+z_1\not{p}-z_2\not{q})\gamma^\mu(\not{l}+z_1\not{p}+(1-z_2)\not{q}) \\
&= -2(\not{l}+z_1\not{p}'-(z_1+z_2)\not{q})\gamma^\mu(\not{l}+z_1\not{p}+(1-z_2)\not{q}) \qquad (7)
\end{aligned}$$

By using the equations of motion, $\bar{u}(p')(\not{p}'-m)=0$, $(\not{p}-m)u(p)=0$,

$$\begin{aligned}
&= -2(\not{l}+z_1m-(1-z_3)\not{q})\gamma^\mu(\not{l}+z_1m+(1-z_2)\not{q}) \\
&= -2[\not{l}\gamma^\mu\not{l}+(z_1m-(1-z_3)\not{q})\gamma^\mu(z_1m+(1-z_2)\not{q})] \qquad (8)
\end{aligned}$$

Since the denominator of the integrand is symmetric under $l\to -l$, we dropped the terms linear in l above. Now we use some tricks to simplify the expression.

$$\begin{aligned}
&(z_1m-(1-z_3)\not{q})\gamma^\mu(z_1m+(1-z_2)\not{q}) \\
&= z_1^2m^2\gamma^\mu+z_1m[-(1-z_3)\not{q}\gamma^\mu+(1-z_2)\gamma^\mu\not{q}]-(1-z_3)(1-z_2)\not{q}\gamma^\mu\not{q} \qquad (9)
\end{aligned}$$

The first trick is on terms in the square bracket:

$$\begin{aligned}
&-(1-z_3)\not{q}\gamma^\mu+(1-z_2)\gamma^\mu\not{q} \\
&= -(1-z_3)\frac{1}{2}(\{\not{q},\gamma^\mu\}+[\not{q},\gamma^\mu])+(1-z_2)\frac{1}{2}(\{\not{q},\gamma^\mu-[\not{q},\gamma^\mu]) \\
&= (z_3-z_2)\frac{1}{2}\{\not{q},\gamma^\mu\}-(2-z_2-z_3)\frac{1}{2}[\not{q},\gamma^\mu] \\
&= (z_3-z_2)q^\mu-(1+z_1)\frac{1}{2}q_\nu[\gamma^\nu,\gamma^\mu] \qquad (10)
\end{aligned}$$

By using the definition of $\sigma^{\mu\nu}=\frac{i}{2}[\gamma^\mu,\gamma^\nu]$, we obtain

$$-(1-z_3)\not{q}\gamma^\mu+(1-z_2)\gamma^\mu\not{q}=(z_3-z_2)q^\mu-(1+z_1)i\sigma^{\mu\nu}q_\nu. \qquad (11)$$

The last term is

$$\begin{aligned}
\not{q}\gamma^\mu\not{q} &= \not{q}(-\not{q}\gamma^\mu+\{\gamma^\mu,\not{q}\}) \\
&= -q^2\gamma^\mu+2\not{q}q^\mu \qquad (12)
\end{aligned}$$

But recalling the fact that $\bar{u}(p')\not{q}u(p)=\bar{u}(p')(\not{p}'-\not{p})u(p)=\bar{u}(p')(m-m)u(p)=0$, it is simply

$$\not{q}\gamma^\mu\not{q}=-q^2\gamma^\mu \qquad (13)$$

By putting them together, the first term in the numerator is given by

$$-2\not{k}\gamma^\mu\not{k}'=-2[\not{l}\gamma^\mu\not{l}+z_1^2m^2\gamma^\mu+z_1m((z_3-z_2)q^\mu-(1+z_1)i\sigma^{\mu\nu}q_\nu)+(1-z_2)(1-z_3)q^2\gamma^\mu] \quad (14)$$

The second term in the numerator is rewritten using the Gordon identity, $\bar{u}(p')(p+p')^\mu u(p) = \bar{u}(p')(2m\gamma^\mu - i\sigma^{\mu\nu}q_\nu)u(p)$. We obtain,

$$\begin{aligned}
4m(k+k')^\mu &= 4m(l + z_1 p - z_2 q + l + z_1 p + (1-z_2)q)^\mu \\
&= 4m(2l + z_1 p' - (z_1+z_2)q + z_1 p + (1-z_2)q)^\mu \\
&= 4m(2l + z_1(p'+p) + (z_3-z_2)q)^\mu \\
&= 4m(z_1(2m\gamma^\mu - i\sigma^{\mu\nu}q_\nu) + (z_3-z_2)q^\mu) \qquad (15)
\end{aligned}$$

In the last equality, we dropped a term linear in l because it vanishes upon integration over d^4l.

Now we can add all terms in the numerator of the integrand.

$$\begin{aligned}
&-2\not{k}\gamma^\mu\not{k}' + 4m(k+k')^\mu - 2m^2\gamma^\mu \\
&= -2[\not{l}\gamma^\mu\not{l} + z_1^2 m^2\gamma^\mu + z_1 m((z_3-z_2)q^\mu - (1+z_1)i\sigma^{\mu\nu}q_\nu) + (1-z_2)(1-z_3)q^2\gamma^\mu] \\
&\quad +4m(z_1(2m\gamma^\mu - i\sigma^{\mu\nu}q_\nu) + (z_3-z_2)q^\mu) - 2m^2\gamma^\mu \\
&= -2\not{l}\gamma^\mu\not{l} + \gamma^\mu(2(-z_1^2 + 4z_1 - 1)m^2 - 2(1-z_2)(1-z_3)q^2) \\
&\quad -2mz_1(1-z_1)i\sigma^{\mu\nu}q_\nu + 2(2-z_1)(z_3-z_2)mq^\mu \qquad (16)
\end{aligned}$$

The last term is anti-symmetric under the interchange $z_2 \leftrightarrow z_3$ while the denominator and integation measure are symmetric, and hence can be dropped.

Finally, the term $\not{l}\gamma^\mu\not{l}$ can be simplified. Because the integration is done in a Lorentz-invariant fashion, $l_\rho l_\sigma$ can be replaced by $g_{\rho\sigma}l^2/4$. The factor $1/4$ comes from the requirement that you obtain the correct normalization upon contracting ρ and σ indices using the Minkowski metric. Therefore in the integrand,

$$\begin{aligned}
\not{l}\gamma^\mu\not{l} &= l_\rho l_\sigma \gamma^\rho\gamma^\mu\gamma^\sigma \\
&= \frac{1}{4}l^2 g_{\rho\sigma}\gamma^\rho\gamma^\mu\gamma^\sigma \\
&= \frac{1}{4}l^2\gamma^\rho\gamma^\mu\gamma_\rho \\
&= -\frac{1}{2}l^2\gamma^\mu \qquad (17)
\end{aligned}$$

by using the identity $\gamma^\rho\not{a}\gamma_\rho = -2\not{a}$ again. Therefore, the numerator is now given by

$$\begin{aligned}
&-2\not{k}\gamma^\mu\not{k}' + 4m(k+k')^\mu - 2m^2\gamma^\mu \\
&= \gamma^\mu(l^2 + 2(-z_1^2 + 4z_1 - 1)m^2 - 2(1-z_2)(1-z_3)q^2) - 2mz_1(1-z_1)i\sigma^{\mu\nu}q_\nu \qquad (18)
\end{aligned}$$

The vertex function is given by

$$\delta\Gamma^\mu(p',p) = -ie^2 2\int_0^1 d^3z\,\delta(\sum_i^3 z_i - 1)\int\frac{d^4l}{(2\pi)^4}$$
$$\frac{\gamma^\mu(l^2 + 2(-z_1^2+4z_1-1)m^2 - 2(1-z_2)(1-z_3)q^2) - 2mz_1(1-z_1)i\sigma^{\mu\nu}q_\nu}{[l^2 - (1-z_1)^2m^2 - z_1\mu^2 + z_2z_3q^2]^3}. \qquad (19)$$

Since this is a function of q only, we henceforth write it as $\delta\Gamma^\mu(q)$.

Now we evaluate the integrals for $q = 0$. It simplifies drastically to

$$\delta\Gamma^\mu(0) = -ie^2 2\int_0^1 d^3z\delta(\sum_i^3 z_i - 1)\int \frac{d^4l}{(2\pi)^4}\frac{\gamma^\mu(l^2 + 2(-z_1^2 + 4z_1 - 1)m^2)}{[l^2 - (1-z_1)^2m^2 - z_1\mu^2]^3}. \qquad (20)$$

First perfom the l integration. We need to deal with the ultraviolet-divergent piece l^2 and the rest separately. We use the formulae

$$\int \frac{d^4l}{(2\pi)^4}\frac{1}{[l^2 - M^2]^n} = \frac{i(-1)^n}{(4\pi)^2}\frac{1}{(n-2)(n-1)}\frac{1}{(M^2)^{n-2}}, \qquad (21)$$

and

$$\begin{aligned}
&\int \frac{d^4l}{(2\pi)^4}\frac{l^2}{[l^2 - M^2]^n} \\
&= \int \frac{d^4l}{(2\pi)^4}\left(\frac{1}{[l^2 - M^2]^{n-1}} + \frac{M^2}{[l^2 - M^2]^n}\right) \\
&= \frac{i(-1)^{n-1}}{(4\pi)^2}\frac{1}{(n-3)(n-2)}\frac{1}{(M^2)^{n-1}} + \frac{i(-1)^n}{(4\pi)^2}\frac{1}{(n-2)(n-1)}\frac{1}{(M^2)^{n-1}} \\
&= \frac{i(-1)^{n-1}}{(4\pi)^2}\frac{2}{(n-3)(n-2)(n-1)}\frac{1}{(M^2)^{n-1}}. \qquad (22)
\end{aligned}$$

The ultraviolet-divergence piece must be regularized by modifying the photon propagator

$$\frac{1}{(p-k)^2 - \mu^2} \to \frac{1}{(p-k)^2 - \mu^2} - \frac{1}{(p-k)^2 - \Lambda^2} = \frac{1}{(p-k)^2 - \mu^2}\frac{\mu^2 - \Lambda^2}{(p-k)^2 - \Lambda^2}. \qquad (23)$$

Note that $(p-k)^2 = ((1-z_1)p - l + z_2q)^2 = ((1-z_1)p - l)^2$ for $q = 0$. Since the momentum dependence becomes important only when $l \simeq \Lambda$, $p \simeq m$ in the bracket is completely neglibible. Therefore, it sufficies to introduce a factor $(\mu^2 - \Lambda^2)/(l^2 - \Lambda^2)$ in the above integral. We also utilize the following formula,

$$\begin{aligned}
\frac{1}{A^nB} &= \frac{(-1)^{n-1}}{(n-1)!}\left(\frac{\partial}{\partial A}\right)^{n-1}\frac{1}{AB} \\
&= \frac{(-1)^{n-1}}{(n-1)!}\left(\frac{\partial}{\partial A}\right)^{n-1}\int_0^1 dx\frac{1}{[xA + (1-x)B]^2} \\
&= \frac{(-1)^{n-1}}{(n-1)!}\int_0^1 dx\frac{(-1)^{n-1}n!x^{n-1}}{[xA + (1-x)B]^{n+1}} \\
&= \int_0^1 dx\frac{nx^{n-1}}{[xA + (1-x)B]^{n+1}}. \qquad (24)
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\int \frac{d^4l}{(2\pi)^4}\frac{l^2}{[l^2-(1-z_1)^2m^2-z_1\mu^2]^3}\frac{\mu^2-\Lambda^2}{l^2-\Lambda^2} \\
&= \int_0^1 dx \int \frac{d^4l}{(2\pi)^4}\frac{3x^2l^2(\mu^2-\Lambda^2)}{[l^2-x(1-z_1)^2m^2-xz_1\mu^2-(1-x)\Lambda^2]^4} \\
&= \int_0^1 dx \frac{i(-1)^3}{(4\pi)^2}\frac{2}{1\cdot 2\cdot 3}\frac{3x^2(\mu^2-\Lambda^2)}{x(1-z_1)^2m^2+xz_1\mu^2+(1-x)\Lambda^2]^4} \qquad (25)
\end{aligned}
$$

A conventional integration gives

$$
\int_0^1 dx\frac{x^2}{xA+(1-x)B} = \frac{A^2-4AB+3B^2+2B^2\ln(A/B)}{2(A-B)^3}. \qquad (26)
$$

In our case $A=(1-z_1)^2m^2+z_1\mu^2 \ll B=\Lambda^2$, and hence it is well approximated by $(3+2\ln(A/B))/(-2B)$. The above integral reduces to

$$
\begin{aligned}
\text{Eq. (25)} &= \frac{-i}{(4\pi)^2}\frac{1}{3}3(\mu^2-\Lambda^2)\frac{3+2\ln((1-z_1)^2m^2+z_1\mu^2)/\Lambda^2}{-2\Lambda^2} \\
&= \frac{-i}{(4\pi)^2}\frac{1}{2}\left(3+2\ln\frac{(1-z_1)^2m^2+z_1\mu^2}{\Lambda^2}\right) \qquad (27)
\end{aligned}
$$

Its contribution to $\delta\Gamma^\mu(0)$ is given by

$$
\begin{aligned}
&-ie^2 2\int_0^1 d^3z\delta(\sum_i^3 z_i-1)\gamma^\mu\frac{-i}{(4\pi)^2}\frac{1}{2}\left(3+2\ln\frac{(1-z_1)^2m^2+z_1\mu^2}{\Lambda^2}\right) \\
&= \frac{-2e^2}{(4\pi)^2}\gamma^\mu\int_0^1 dz_1\int_0^{1-z_1}dz_2\frac{1}{2}\left(3+2\ln\frac{(1-z_1)^2m^2+z_1\mu^2}{\Lambda^2}\right) \\
&= \frac{-\alpha}{4\pi}\gamma^\mu\int_0^1 dz_1(1-z_1)\left(3+2\ln\frac{(1-z_1)^2m^2+z_1\mu^2}{\Lambda^2}\right) \\
&= \frac{-\alpha}{4\pi}\gamma^\mu\int_0^1 dz_1(1-z_1)\left(3+2\ln\frac{m^2}{\Lambda^2}+4\ln(1-z_1)\right) \\
&= \frac{-\alpha}{4\pi}\gamma^\mu\frac{1}{2}\left(1+2\ln\frac{m^2}{\Lambda^2}\right) \qquad (28)
\end{aligned}
$$

Above, we set $\mu=0$ because it does not cause any infrared singularity.

Now we come back to the ultraviolet-safe part in $\delta\Gamma^\mu(0)$. This piece is not infrared-safe, however. It is given by

$$
\begin{aligned}
&-ie^2 2\int_0^1 d^3z\delta(\sum_i^3 z_i-1)\int\frac{d^4l}{(2\pi)^4}\frac{\gamma^\mu(2(-z_1^2+4z_1-1)m^2)}{[l^2-(1-z_1)^2m^2-z_1\mu^2]^3} \\
&= -ie^2 2\int_0^1 dz_1\int_0^{1-z_1}dz_2\frac{-i}{(4\pi)^2}\frac{1}{2}\frac{\gamma^\mu 2(-z_1^2+4z_1-1)m^2}{(1-z_1)^2m^2+z_1\mu^2} \\
&= \frac{-2\alpha}{4\pi}\gamma^\mu\int_0^1 dz_1(1-z_1)\frac{(-z_1^2+4z_1-1)m^2}{(1-z_1)^2m^2+z_1\mu^2} \qquad (29)
\end{aligned}
$$

By changing the variable $z_1 \to 1 - z$,

$$= \frac{-2\alpha}{4\pi}\gamma^\mu \int_0^1 dz z \frac{2 - 2z - z^2}{z^2 + (1-z)\mu^2/m^2} \tag{30}$$

The integral is logarithmically divergent at $z \to 0$ if $\mu = 0$. A conventional integration gives

$$\begin{aligned}
&\int_0^1 dz z \frac{2 - 2z - z^2}{z^2 + (1-z)\epsilon} \\
&= -\frac{5}{2} - \epsilon + \frac{\epsilon\,(6 + \epsilon - \epsilon^2)\,\arctan((2-\epsilon)/\sqrt{(4-\epsilon)\,\epsilon})}{\sqrt{(4-\epsilon)\,\epsilon}} \\
&\quad + \frac{\epsilon\,(6 + \epsilon - \epsilon^2)\,\arctan(\epsilon/\sqrt{(4-\epsilon)\,\epsilon})}{\sqrt{(4-\epsilon)\,\epsilon}} + \frac{(-2 + \epsilon + \epsilon^2)\,\log(\epsilon)}{2}
\end{aligned} \tag{31}$$

In our case, $\epsilon = \mu^2/m^2 \ll 1$, and

$$\simeq \left(-\frac{5}{2} - \log(\epsilon)\right) \tag{32}$$

Therefore, this contribution to $\delta\Gamma^\mu(0)$ is

$$\frac{-2\alpha}{4\pi}\gamma^\mu \left(-\frac{5}{2} - \ln\frac{\mu^2}{m^2}\right) \tag{33}$$

By combining two pieces in $\delta\Gamma^\mu(0)$, we obtain

$$\begin{aligned}
\delta\Gamma^\mu(0) &= \frac{-\alpha}{4\pi}\gamma^\mu \frac{1}{2}\left(1 + 2\ln\frac{m^2}{\Lambda^2}\right) + \frac{-2\alpha}{4\pi}\gamma^\mu \left(-\frac{5}{2} - \ln\frac{\mu^2}{m^2}\right) \\
&= \frac{-\alpha}{4\pi}\gamma^\mu \left(\frac{1}{2} + \ln\frac{m^2}{\Lambda^2} - 5 + 2\ln\frac{m^2}{\mu^2}\right) \\
&= \frac{\alpha}{4\pi}\gamma^\mu \left(\ln\frac{\Lambda^2}{m^2} + \frac{9}{2} - 2\ln\frac{m^2}{\mu^2}\right).
\end{aligned} \tag{34}$$

Recall that

$$Z_2 = 1 - \frac{\alpha}{4\pi}\gamma^\mu \left(\ln\frac{\Lambda^2}{m^2} + \frac{9}{2} - 2\ln\frac{m^2}{\mu^2}\right) \tag{35}$$

and hence

$$(Z_2 - 1)\gamma^\mu = -\delta\Gamma^\mu(0). \tag{36}$$

(Often, people define $\Gamma^\mu(0) = Z_1^{-1}\gamma^\mu$, and the above relation is expressed as $Z_2 = Z_1$.) The renormalized vertex function is therefore given by

$$\begin{aligned}
\sqrt{Z_2}\sqrt{Z_2}\Gamma^\mu(q) &= Z_2\Gamma^\mu(q) \\
&= (1 + (Z_2 - 1))(\gamma^\mu + \delta\Gamma^\mu(q)) \\
&= \gamma^\mu + (Z_2 - 1)\gamma^\mu + \delta\Gamma^\mu(q) + O(\alpha/4\pi)^2 \\
&= \gamma^\mu - \delta\Gamma^\mu(0) + \delta\Gamma^\mu(q) + O(\alpha/4\pi)^2 \\
&= \gamma^\mu + (\delta\Gamma^\mu(q) - \delta\Gamma^\mu(0)).
\end{aligned} \tag{37}$$

The final stage is to show that the combination $\delta\Gamma^\mu(q) - \delta\Gamma^\mu(0)$ is indeed ultraviolet safe. By defining

$$Z_2\Gamma^\mu(q) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2), \tag{38}$$

and going back to the integral form of Γ^μ, we find

$$\begin{aligned} F_1(q^2) - 1 &= -ie^2 2\int_0^1 d^3z\delta(\sum_i^3 z_i - 1)\int \frac{d^4l}{(2\pi)^4} \\ &\left(\frac{l^2 + 2(-z_1^2 + 4z_1 - 1)m^2 - 2(1-z_2)(1-z_3)q^2}{[l^2 - (1-z_1)^2m^2 - z_1\mu^2 + z_2z_3q^2]^3} - (q^2 \to 0)\right), \end{aligned} \tag{39}$$

$$F_2(q^2) = -ie^2 2\int_0^1 d^3z\delta(\sum_i^3 z_i - 1)\int \frac{d^4l}{(2\pi)^4}\frac{-2mz_1(1-z_1)\cdot 2m}{[l^2 - (1-z_1)^2m^2 - z_1\mu^2 + z_2z_3q^2]^3}. \tag{40}$$

To prove F_1 is finite, let us first concentrate on the term with l^2. By using the Wick rotation of the l^0 integral,

$$\begin{aligned} &\int \frac{d^4l}{(2\pi)^4}\left(\frac{l^2}{[l^2 - (1-z_1)^2m^2 - z_1\mu^2 + z_2z_3q^2]^3} - \frac{l^2}{[l^2 - (1-z_1)^2m^2 - z_1\mu^2]^3}\right) \\ &= i\int \frac{d^4l_E}{(2\pi)^4}\left(\frac{-l_E^2}{[-l_E^2 - (1-z_1)^2m^2 - z_1\mu^2 + z_2z_3q^2]^3} - (q^2 \to 0)\right) \\ &= i\int_0^\infty \frac{\pi^2 l_E^2 dl_E^2}{(2\pi)^4}\left(\frac{l_E^2}{[l_E^2 + (1-z_1)^2m^2 + z_1\mu^2 - z_2z_3q^2]^3} - (q^2 \to 0)\right) \\ &= \frac{i}{(4\pi)^2}\left\{\left[-\frac{1}{2}\frac{(l_E^2)^2}{[l_E^2 + (1-z_1)^2m^2 + z_1\mu^2 - z_2z_3q^2]^2} - (q^2 \to 0)\right]_0^\infty \right. \\ &\qquad \left. + \int_0^\infty dl_E^2\left(\frac{l_E^2}{[l_E^2 + (1-z_1)^2m^2 + z_1\mu^2 - z_2z_3q^2]^2} - (q^2 \to 0)\right)\right\} \\ &= \frac{i}{(4\pi)^2}\left\{\left[-\frac{l_E^2}{l_E^2 + (1-z_1)^2m^2 + z_1\mu^2 - z_2z_3q^2} - (q^2 \to 0)\right]_0^\infty \right. \\ &\qquad \left. + \int_0^\infty dl_E^2\left(\frac{1}{l_E^2 + (1-z_1)^2m^2 + z_1\mu^2 - z_2z_3q^2} - (q^2 \to 0)\right)\right\} \\ &= \frac{i}{(4\pi)^2}\ln\frac{(1-z_1)^2m^2 + z_1\mu^2}{(1-z_1)^2m^2 + z_1\mu^2 - z_2z_3q^2} \end{aligned} \tag{41}$$

All other terms are ultraviolet safe even before the subtraction. In the end we obtain,

$$\begin{aligned} &F_1(q^2) - F_1(0) \\ &= \frac{\alpha}{4\pi}2\int_0^1 d^3z\delta(\sum_i^3 z_i - 1)\left\{\ln\frac{z_1\mu^2 + (1-z_1)^2m^2}{z_1\mu^2 + (1-z_1)^2m^2 - z_2z_3q^2}\right. \end{aligned}$$

$$
-\frac{1}{2}\frac{2(-1+4z_1-z_1^2)m^2-2(1-z_2)(1-z_3)q^2}{z_1\mu^2+(1-z_1)^2m^2-z_2z_3q^2}+\frac{1}{2}\frac{2(-1+4z_1+z_1^2)m^2}{z_1\mu^2+(1-z_1)^2m^2}\Bigg\} \quad (42)
$$

$$
F_2(q^2) = \frac{\alpha}{4\pi}2\int_0^1 d^3z\delta(\sum_i^3 z_i-1)\frac{-1}{2}\frac{-2z_1(1-z_1)m\cdot 2m}{z_1\mu^2+(1-z_1)^2m^2-z_2z_3q^2} \quad (43)
$$

The results are finite and agree with the Eqs. (6.56), (6.57) in the textbook.