HW #8, due Oct 29

1. Trace technique. Calculate the spin-summed squared amplitude for the $e^+e^- \to \mu^+\mu^-$ using the trace technique. Assume the electron is massles, but retain the muon mass m.

(a) Take the complex conjugate $\mathcal{M}^*$ using

$$i\mathcal{M} = ie^2 \frac{1}{s}[\bar{u}(k)\gamma^\mu v(\bar{k})][\bar{v}(\bar{p})\gamma_\mu u(k)]. \tag{1}$$

(b) Multiply $\mathcal{M}$ and $\mathcal{M}^*$ and rewrite the product in terms of traces. Use $\sum_\pm u_\pm(p)\bar{u}_\pm(p) = \not{p} + m$ and $\sum_\pm v_\pm(p)\bar{v}_\pm(p) = \not{p} - m$.

(c) Evaluate the traces.

(d) Show that

$$\sum_{\text{helicities}} |\mathcal{M}|^2 = 4e^4 \left(1 + \cos^2\theta + \frac{m^2}{E^2}\sin^2\theta\right) \tag{2}$$

(e) Calculate the total cross section by using the formula

$$\sigma = \frac{1}{2s}\frac{1}{2}\frac{1}{2}\frac{\bar{\beta}_f}{8\pi}\int_{-1}^{1}\frac{d\cos\theta}{2}\sum_{\text{helicites}}|\mathcal{M}|^2 \tag{3}$$

Here, two factors of 1/2 come from the spin average of initial electron and positron. Express it in terms of $\alpha = e^2/4\pi$, $\beta = \bar{\beta}_f$ (remember this is $m_1 = m_2$ case), and s.

(f) Evaluate the numerical value of the cross section and show that

$$\sigma = \frac{86.8 \text{ nb}}{(\sqrt{s}/\text{GeV})^2} \tag{4}$$

(g) TRISTAN accelerator ran at center-of-energy of about 60 GeV and with a luminosity of $10^{31}\text{cm}^{-2}\text{s}^{-1}$. How long time one has to wait for another muon pair on average?

2. Coulomb scattering. We study the scattering of electron in a static Coulomb potential. The static Coulomb potential is given by

$$A^0 = \frac{-Ze}{4\pi|\vec{x}|}, \tag{5}$$

and then the interaction Hamiltonian is given by

$$H_{int} = e(\bar{e}\gamma^\mu e)A_\mu = \frac{-Z\alpha}{|\vec{x}|}\bar{e}\gamma^0 e. \tag{6}$$

(a) Use the LSZ formula to write down the matrix element $iT_{fi} = \langle e, \vec{p}'|iT|e, \vec{p}\rangle$ in terms of the Heisenberg operators.

(b) Calculate the correlation function $\langle 0|Te(x)\bar{e}(y)(-i)\int d^4z H_I(z)|0\rangle$.

(c) Combining the above two, show that the matrix element is given by

$$iT_{fi} = iZ\alpha \frac{4\pi \bar{u}(p')\gamma^0 u(p)}{|\vec{q}|^2} 2\pi\delta(E' - E) \tag{7}$$

where $\vec{q} = \vec{p} - \vec{p}'$. Note that the three-momentum is not conserved because of the presence of the static Coulomb potential which breaks the translational invariance. The following Fourier transform is useful to know:

$$\int \frac{d^3x}{|\vec{x}|} e^{-i\vec{q}\cdot\vec{x}} = \frac{4\pi}{|\vec{q}|^2}. \tag{8}$$

(d) Show that

$$\sum_{\text{helicities}} |\bar{u}(p')\gamma^0 u(p)|^2 = \mathrm{Tr}(\not{p}' + m_e)\gamma^0(\not{p} + m_e)\gamma^0 = 8E^2\left(1 - \beta^2 \sin^2\frac{\theta}{2}\right). \tag{9}$$

(e) The probability of scattering is given by

$$P = \frac{1}{2}\frac{1}{2EV}\frac{1}{2E'V}\int \frac{V d^3p'}{(2\pi)^3}|iT_{fi}|^2, \tag{10}$$

where the factor of 1/2 is from the spin average for the initial electron. The cross section is then given by $\sigma = P/L$ with $L = \beta T/V$. Using $[2\pi\delta(E' - E)]^2 = 2\pi\delta(E' - E)T = 2\pi\frac{E'}{p'}\delta(p' - p)T$, perform dp' integral in $d^3p' = p'^2 dp' d\Omega$ and obtain

$$\frac{d\sigma}{d\Omega} = \frac{(Z\alpha)^2}{4p^2\beta^2\sin^4\theta/2}\left(1 - \beta^2\sin^2\frac{\theta}{2}\right). \tag{11}$$

This is called Mott formula.

(f) Verify that the non-relativistic limit $\beta \to 0$ agrees with the Rutherford formula.