HW #7, due Oct 22

1. The Lorentz-invariant phase space. The Lorentz-invariant phase space is

$$d\Phi_n = \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 E_{p_i}} (2\pi)^4 \delta^4 (\sum_{i=1}^{n} p_i - q) \tag{1}$$

where q is the total four-momentum of the system. For a collision process, q is given by the sum of four-momenta of the two incoming particles, $q = q_1 + q_2$. It is always convenient to go to the center-of-momentum frame where $q = (\sqrt{s}, 0, 0, 0)$.

(a) Let us consider the two-body phase space with final state particles with masses m_1 and m_2. Show that the energies and momenta of the final state particles are given by

$$E_1 = \frac{\sqrt{s}}{2}\left(1 + \frac{m_1^2}{s} - \frac{m_2^2}{s}\right) \tag{2}$$

$$E_2 = \frac{\sqrt{s}}{2}\left(1 - \frac{m_1^2}{s} + \frac{m_2^2}{s}\right) \tag{3}$$

$$|\vec{p}_1| = |\vec{p}_2| = \frac{\sqrt{s}}{2}\bar{\beta}_f \tag{4}$$

$$\bar{\beta}_f = \sqrt{1 - 2\frac{m_1^2 + m_2^2}{s} + \left(\frac{m_1^2 - m_2^2}{s}\right)^2}. \tag{5}$$

(b) We would like to work out the two-body phase space explicity in terms of polar angle θ, azimuthal angle ϕ and the masses of the final state particles m_1 and m_2. Show that the two-body phase space can be rewritten as

$$d\Phi_2 = \frac{\bar{\beta}_f}{8\pi} \frac{d\cos\theta}{2} \frac{d\phi}{2\pi}. \tag{6}$$

(Hint: First do the momentum integration over $\vec{p}_2$. Then write the delta function of energy conservation in terms of $\vec{p}_1$. Rewrite the $\vec{p}_1$ integration using polar coordinates $|\vec{p}_1|$, $\cos\theta$ and ϕ, and integrate over $|\vec{p}_1|$ using the delta function.)

(c) How do the energies, momenta and the $\bar{\beta}_f$ factor simplify for the following two cases? (i) $m_2 = 0$. (ii) $m_1 = m_2 \neq 0$.

2. Helicity amplitudes of $e^+e^- \to \mu^+\mu^-$. The Feynman amplitude of this process is given by

$$i\mathcal{M} = ie^2 \frac{g_{\mu\nu}}{s} [\bar{u}(k)\gamma^\mu v(\bar{k})][\bar{v}(\bar{p})\gamma^\nu u(k)] \tag{7}$$

where k, $\bar{k}$, p, $\bar{p}$ are the four-momenta of μ^-, μ^+, e^-, and e^+, respectively. Assume that both electron and muon are massless. We use the center-of-momentum frame and the four-momenta are given by

$$\begin{aligned} p^\mu &= E(1,0,0,1) && (8)\\ \bar{p}^\mu &= E(1,0,0,-1) && (9)\\ k^\mu &= E(1,\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta) && (10)\\ \bar{k}^\mu &= E(1,-\sin\theta\cos\phi,-\sin\theta\sin\phi,-\cos\theta) && (11) \end{aligned}$$

Use explicit solutions to the Dirac equation in the chiral representation (as distributed together with HW #3), not Pauli–Dirac representation, as it is easier for this purpose.

(a) Convince yourself that the spinor product of the muons vanish for the helicity combination $\mu_L^-\mu_L^+$. (Recall that the left-handed state has helicity $-1/2$, and is represented either by u_- or v_+ spinor.) The same is true for $\mu_R^-\mu_R^+$ combination.

(b) Write explicit four-vectors for $[\bar{v}(\bar{p})\gamma^\nu u(k)]$ for helicity combinations $e_L^-e_R^+$ and $e_R^-e_L^+$ separately. Here, the positron helicity spinors are given by $\theta=\pi$ and $\phi=0$.

(c) Write explicit four-vectors for $[\bar{u}(k)\gamma^\mu v(\bar{k})]$ for helicity combinations $e_L^-e_R^+$ and $\mu_R^-\mu_L^+$ separately. Here, the μ^+ helicity spinors are obtained by substituting θ by $\pi-\theta$, and ϕ by $\phi+\pi$.

(d) Multiply them together to obtain the following helicity amplitudes:

$$\begin{aligned} \mathcal{M}_{RL\to RL} &= e^2(1+\cos\theta)e^{i\phi} && (12)\\ \mathcal{M}_{RL\to LR} &= -e^2(1-\cos\theta)e^{i\phi} && (13)\\ \mathcal{M}_{LR\to RL} &= -e^2(1-\cos\theta)e^{-i\phi} && (14)\\ \mathcal{M}_{LR\to LR} &= e^2(1+\cos\theta)e^{-i\phi} && (15) \end{aligned}$$

(e) Suppose you have a beam of purely right-handed electrons. Select muons produced in the forward region $\cos\theta>0$. What fraction of the muons is right-handed?

(f) Calculate the spin-summed squared amplitude $\sum_{\text{helicities}}|\mathcal{M}|^2$.