HW #6, due Oct 15

1. Toy Dirac Model, Wick's theorem, LSZ reduction formula. Consider the following quantum mechanics Lagrangian,

$$L = \bar{\psi}(i\sigma_3\partial_t - m)\psi, \tag{1}$$

where σ_3 is a Pauli matrix, and $\bar{\psi}$ is defined by $\bar{\psi} = \psi^\dagger\sigma_3$. ψ is a two-component variable. We quantize the dynamical variable ψ and its canonical conjugate momentum $i\psi^\dagger$ using the canonical anti-commutation relation $\{\psi_\alpha, i\psi^\dagger_\beta\} = i\delta_{\alpha\beta}$ for $\alpha, \beta = 1, 2$.

(1) Show that the equation of motion $(i\sigma_3\partial_t - m)\psi = 0$ has a positive and a negative energy solution,

$$(i\sigma_3\partial_t - m)ue^{-imt} = 0, \qquad u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2}$$

$$(i\sigma_3\partial_t - m)ve^{imt} = 0, \qquad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3}$$

(2) We expand the operator ψ as

$$\psi(t) = aue^{-imt} + b^\dagger ve^{imt} \tag{4}$$

Show that the operators a and b satisfy the algebra of creation and annihilation operators, $\{a, a^\dagger\} = \{b, b^\dagger\} = 1$, using the canonical anti-commutation relation $\{\psi_\alpha, \psi^\dagger_\beta\} = \delta_{\alpha\beta}$ where $\alpha, \beta = 1, 2$.

note The Hilbert space consists of four states, the "vacuum" $|0\rangle$ defined by $a|0\rangle = b|0\rangle = 0$, "one-particle states" $|a\rangle = a^\dagger|0\rangle$, $|b\rangle = b^\dagger|0\rangle$ and the "pair state" $|ab\rangle = a^\dagger b^\dagger|0\rangle$.

(3) Show that the Hamiltonian is given by $H_0 = \psi^\dagger m\sigma_3\psi = m(a^\dagger a - bb^\dagger) = m(a^\dagger a + b^\dagger b) + \text{constant}$.

(4) Show that the Feynman propagator is given by

$$\begin{aligned}
S_F(t_1-t_2) &= \langle 0|T\psi_\alpha(t_1)\bar{\psi}_\beta(t_2)|0\rangle \\
&= \theta(t_1-t_2)e^{-im(t_1-t_2)}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}_{\alpha\beta} + \theta(t_2-t_1)e^{im(t_1-t_2)}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}_{\alpha\beta} \\
&= \int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{i(E\sigma_3+m)_{\alpha\beta}}{E^2-m^2+i\epsilon}e^{-iE(t_1-t_2)} \qquad (5)
\end{aligned}$$

(Hint: Consider two cases $t_1 > t_2$ or $t_1 < t_2$ separately, and use contour integral in lower or upper half plane, respectively. To locate the poles, the figure on p. 31 of the textbook may help.)

note The above Feynman propagator is often written as

$$S_F(t_1-t_2) = \int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{ie^{-iE(t_1-t_2)}}{E\sigma_3-m+i\epsilon} \qquad (6)$$

(5) Calculate $\langle 0|T\psi(t_1)\bar{\psi}(t_2)\psi(t_3)\bar{\psi}(t_4)|0\rangle$ when $t_1 > t_2 > t_3 > t_4$ in two ways. (1) Use Wick's theorem. (2) Work it out explicitly using annihilation and creation operators.

(6) Now we add a time-dependent perturbation to the system:

$$H = H_0 + V, \qquad V = f(t)\psi^\dagger\sigma_1\psi \qquad (7)$$

where $f(t)$ is a c-number function of time, and assume $f(t) \to 0$ for both the inifinite past $t < -T$ and the inifinite future $t > T$. Calculate the amplitude $\langle 0(\infty)|0(-\infty)\rangle = \langle 0|_I T e^{-i\int_{-\infty}^{\infty}V_I(t)dt}|0\rangle_I$ at second order in perturbation using the Wick's theorem.

note Because of the perturbation, the Heisenberg operator $\psi(t)$ does not follow the equation of motion of the free field. However, it does follow the free equation for $t < -T$ and $t > T$ because $f(t)$ vanishes. Therefore, the following Heisenberg operator

$$e^{imt}\bar{u}\psi(t) \qquad (8)$$

reduces to the annihilation operator a for both $t < -T$ and $t > T$. Similarly, another Heisenberg operator

$$e^{-imt}\bar{\psi}(t)v \qquad (9)$$

reduces to $-b$ in these limits.

(7) Consider the following matrix element in the Heisenberg picture

$$\int_{-\infty}^{\infty} dte^{imt}(-i)\bar{u}(i\sigma_3\partial_t - m)\langle 0|T\mathcal{O}(t_1)\psi(t)|0\rangle \tag{10}$$

where $\mathcal{O}(t_1)$ is an arbitrary (bosonic) Heisenberg operator. Using partial integration, show that it can be simplified to

$$\langle 0|a(t=\infty)\mathcal{O}(t_1) - \mathcal{O}(t_1)a(t=-\infty)|0\rangle = {}_{\rm out}\langle a|\mathcal{O}(t_1)|0\rangle. \tag{11}$$

(8) Consider the following matrix element in the Heisenberg picture

$$\int_{-\infty}^{\infty} dt\langle 0|T\mathcal{O}(t_1)\bar{\psi}_\beta(t)|0\rangle[i(-i\sigma_3\overleftarrow{\partial}_t - m)v]_\beta e^{imt} \tag{12}$$

where $\mathcal{O}(t_1)$ is an arbitrary (bosonic) Heisenberg operator. Using partial integration, show that it can be simplified to

$$\langle 0|(b\mathcal{O}(t_1) - \mathcal{O}(t_1)a)|0\rangle = {}_{\rm out}\langle b|\mathcal{O}(t_1)|0\rangle. \tag{13}$$

(9) Show that the "pair-creation" amplitude ${}_{\rm out}\langle ab|0\rangle_{\rm in}$ can be rewritten as

$$\begin{aligned} {}_{\rm out}\langle ab|0\rangle_{\rm in} &= \int_{-\infty}^{\infty} dt_1 dt_2 e^{imt_1}\left[-i\bar{u}(i\sigma_3\partial_{t_1} - m)\right]_\alpha \\ &\quad \langle 0|T\psi_\alpha(t_1)\bar{\psi}_\beta(t_2)|0\rangle[i(-i\sigma_3\overleftarrow{\partial}_{t_2} - m)v]_\beta e^{imt_2} \end{aligned} \tag{14}$$

note Once we have the above expression for the amplitude, we can calculate the correlation function $\langle 0|T\psi_\alpha(t_1)\bar{\psi}_\beta(t_2)|0\rangle$ using the time-dependent perturbation theory to obtain a perturbative result for the amplitude.

LSZ reduction formula and cross sections

1. Real Klein–Gordon field

$$\begin{aligned}
&\left(\sqrt{Z}\right)^{n+m} {}_{\rm out}\langle p_1, \cdots, p_n | q_1, \cdots q_m\rangle_{\rm in} \\
&= \prod_i^n \int d^4x_i e^{ip_ix_i} i(\Box_{x_i} + m^2) \prod_j^m \int d^4y_j e^{-iq_jy_j} i(\Box_{y_j} + m^2) \\
&\quad \langle 0|T\phi(x_1)\cdots\phi(x_n)\phi(y_1)\cdots\phi(y_m)|0\rangle \qquad (1)
\end{aligned}$$

2. Dirac field

$$\sqrt{Z}|\text{particle}(p,\pm)\rangle_{\rm in} = \int d^4x e^{-ipx} T\bar{\psi}(x)|0\rangle(-i)(-i\overleftarrow{\partial\!\!\!/} - m)u_\pm(p) \qquad (2)$$

$$\sqrt{Z}|\text{anti-particle}(p,\pm)\rangle_{\rm in} = \int d^4x e^{-ipx} \bar{v}_\mp(p) i(i\partial\!\!\!/ - m)T\psi(x)|0\rangle \qquad (3)$$

$$\sqrt{Z}{}_{\rm out}\langle\text{particle}(p,\pm)| = \int d^4x e^{ipx} \bar{u}_\pm(p)(-i)(i\partial\!\!\!/ - m)\langle 0|T\psi(x) \qquad (4)$$

$$\sqrt{Z}{}_{\rm out}\langle\text{anti-particle}(p,\pm)| = \int d^4x e^{ipx} \langle 0|T\bar{\psi}(x) i(-i\overleftarrow{\partial\!\!\!/} - m)v_\mp(p) \qquad (5)$$

Repeated application of above formulae can convert all (anti)-particles in initial or final states into the field operators so that the time-dependent perturbation theory allows you to work out amplitudes.

3. Cross sections

$$i\mathcal{M}(2\pi)^4\delta^4(\sum_i p_i - q_1 - q_2) = {}_{\rm out}\langle p_1, \cdots, p_n|q_1, q_2\rangle_{\rm in} \qquad (6)$$

$$\sigma = \frac{1}{2s\bar{\beta}} \prod_i^n \int d\tilde{p}_i |\mathcal{M}|^2 (2\pi)^4 \delta^4(\sum_i p_i - q_1 - q_2) \qquad (7)$$

where

$$\int d\tilde{p} = \int \frac{d^3p}{(2\pi)^3 2E_p} \qquad (8)$$

$$s = (q_1 + q_2)^2 \qquad (9)$$

$$\bar{\beta}_i = \sqrt{1 - \frac{2(m_1^2 + m_2^2)}{s} + \left(\frac{m_1^2 - m_2^2}{s}\right)^2}. \qquad (10)$$

Here, $m_1^2 = q_1^2$ and $m_2^2 = q_2^2$ are the mass squareds of particles in the initial state.

An Explicit Example: $e^+e^- \to \mu^+\mu^-$

We would like to calculate the cross section of the process $e^+e^- \to \mu^+\mu^-$. The algorithm is the following. (1) Write the matrix element ${}_{\rm out}\langle\mu^+\mu^-|e^+e^-\rangle_{\rm in}$ in terms of Heisenberg field operators using the LSZ reduction formula. (2) Rewrite the correlation function of Heisenberg field operators in terms of field operators in the interaction picture. (3) Calculate the correlation function of the field operators in the interaction picture using the Wick's theorm. (4) Evaluate the amplitude. (5) Stick the amplitude into the formula of the cross section. (6) If you are computing higher order corrections, compute the two-point functions to calculate $\sqrt{Z}$ factors. Below, we discuss only the leading order result so that we can set all $\sqrt{Z} = 1$.

Let me show each of the steps briefly. The interaction Hamiltonian in QED is given by

$$H_{int} = e \int d^4x (\bar{e}\gamma^\mu e + \bar{\mu}\gamma^\mu \mu) A_\mu. \tag{1}$$

(1) LSZ reduction formula

$$\begin{aligned}
&\left(\sqrt{Z_\mu}\right)^2 \left(\sqrt{Z_e}\right)^2 {}_{\rm out}\langle \mu^-(p_1,h_1)\mu^+(p_2,h_2)|e^-(p_3,h_3)e^+(p_4,h_4)\rangle_{\rm in} \\
&= \prod_i^4 \int d^4x_i e^{i(p_1x_1+p_2x_2-p_3x_3-p_4x_4)} \\
&\quad [\bar{u}_{h_1}(p_1)(-i)(i\,\partial\!\!\!/_{x_1} - m_\mu)][\bar{v}_{-h_4}(p_4)i(i\,\partial\!\!\!/_{x_4} - m_e)] \\
&\quad\quad \langle 0|T\mu(x_1)\bar{\mu}(x_2)\bar{e}(x_3)e(x_4)|0\rangle \\
&\quad [i(-i\overleftarrow{\partial\!\!\!/}_{x_2} - m_\mu)v_{-h_2}(p_2)][(-i)(-i\overleftarrow{\partial\!\!\!/}_{x_3} - m_e)u_{h_3}(p_3)]
\end{aligned} \tag{2}$$

(2) Interaction picture.

$$\langle 0|T\mu(x_1)\bar{\mu}(x_2)\bar{e}(x_3)e(x_4)|0\rangle = \frac{\langle 0|T\mu_I(x_1)\bar{\mu}_I(x_2)\bar{e}_I(x_3)e_I(x_4)e^{-i\int d^4y H_I(y)}|0\rangle}{\langle 0|Te^{-i\int d^4y H_I(y)}|0\rangle} \tag{3}$$

(3) Wick's theorem

$$\begin{aligned}
&\prod_i^4 \int d^4x_i e^{i(p_1x_1+p_2x_2-p_3x_3-p_4x_4)} \langle 0|T\mu(x_1)\bar{\mu}(x_2)\bar{e}(x_3)e(x_4)|0\rangle \\
&= \prod_i^4 \int d^4x_i e^{i(p_1x_1+p_2x_2-p_3x_3-p_4x_4)} (-ie)^2 \frac{1}{2!} \int d^4y_1 d^4y_2
\end{aligned}$$

$$
\begin{aligned}
&\langle 0|T\mu_I(x_1)\bar{\mu}_I(x_2)\bar{e}_I(x_3)e_I(x_4)H_I(y_1)H_I(y_2)|0\rangle \\
&-(-ie)^2\frac{1}{2!}\int d^4y_1 d^4y_2 \langle 0|H_I(y_1)H_I(y_2)|0\rangle + O(e)^4 \\
= &\left(\frac{i}{\not{p}_1 - m_\mu}\gamma^\mu \frac{i}{-\not{p}_2 - m_\mu}\right)\left(\frac{i}{-\not{p}_4 - m_e}\gamma^\nu \frac{i}{\not{p}_3 - m_e}\right) \\
&\frac{-ig_{\mu\nu}}{q^2 + i\epsilon}(2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4) + O(e)^4 \qquad (4)
\end{aligned}
$$

Here, I took the Feynman gauge $\xi = 1$ for the photon propagator. Hereafter, I drop $O(e)^4$. $q = (p_1 + p_2) = (p_3 + p_4)$ is the four-momentum in the photon propagator.

(4) Amplitude. Note that the LSZ reduction formula precisely cancels the extra fermion propagators.

$$
i\mathcal{M} = (-ie)^2 \left[\bar{u}_{h_1}(p_1)\gamma^\mu v_{-h_2}(p_2)\right]\left[\bar{v}_{-h_4}(p_4)\gamma^\nu u_{h_3}(p_3)\right]\frac{-ig_{\mu\nu}}{(p_1 + p_2)^2 + i\epsilon} \qquad (5)
$$

Then one can plug in the explicit forms of u and v spinors to obtain the amplitude.

(5) Cross section.

$$
\sigma = \frac{1}{2s\bar{\beta}_i}\int d\tilde{p}_1 d\tilde{p}_2 |\mathcal{M}|^2 (2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4). \qquad (6)
$$

In the center of momentum frame, the phase space integral can be drastically simplified:

$$
\int d\tilde{p}_1 d\tilde{p}_2 (2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4) = \frac{\bar{\beta}_f}{8\pi}\int_{-1}^{1}\frac{d\cos\theta}{2}\int_0^{2\pi}\frac{d\phi}{2\pi}, \qquad (7)
$$

where $\bar{\beta}_f$ is defined by the same formula as $\bar{\beta}_i$ except the mass squareds are those of the final state particles. In particular, we have $m_1^2 = m_2^2 = m_\mu^2$ in this case and hence

$$
\bar{\beta}_f = \sqrt{1 - \frac{4m_\mu^2}{s}}. \qquad (8)
$$