Solution Set #5

Problem #1)

(1) Let $\boxed{X = \frac{1}{\sqrt{2w}}(a + a^+) \quad , \quad p = -i\sqrt{\frac{w}{2}}(a - a^+)}$

The inverse is

$$a = \frac{wx + ip}{\sqrt{2w}} \quad , \quad a^+ = \frac{wx - ip}{\sqrt{2w}}$$

And also $[x, p] = i \Rightarrow$

$$[a, a^+] = \frac{[wx + ip, \ wx - ip]}{2w} = \frac{-i\cancel{w}[x,p] + i\cancel{w}[p,x]}{2\cancel{w}}$$

$$= \frac{-i \cdot i + i \cdot (-i)}{2} = 1$$

And Furthermore

$$H = \frac{1}{2}p^2 + \frac{1}{2}w^2 x^2$$

$$= -\frac{1}{2}\frac{w}{2}(a - a^+)^2 + \frac{1}{2}w^{\cancel{2}}\frac{1}{2\cancel{w}}(a + a^+)^2$$

$$= \frac{w}{4}(aa^+ + a^+a + aa^+ + a^+a)$$

$$= \frac{w}{2}(a^+a + aa^+)$$

$$= w(a^+a + \tfrac{1}{2})$$

Now to get the Heisenberg operators, use

$$[a, H] = w[a, a^+a] = w[a, a^+]a = wa \quad , \quad [a^+, H] = -wa^+$$

so $a_H(t) = e^{iHt} a e^{-iHt} \therefore \frac{\partial}{\partial t} a_H(t) = ie^{iHt}[H, a]e^{-iHt} = -iw \, a_H(t)$

$a_H(t=0) = a \therefore \boxed{a_H(t) = ae^{-iwt}}$

Likewise $\frac{\partial}{\partial t} a_H^+(t) = ie^{iHt}[H, a^+]e^{-iHt} = iw \, a_H^+(t) \therefore \boxed{a_H^+(t) = a^+e^{iwt}}$

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Putting this all together

$$\boxed{\begin{aligned} x(t) &= \frac{1}{\sqrt{2w}}\left(a e^{-iwt} + a^\dagger e^{iwt}\right) \\ p(t) &= -i\sqrt{\frac{w}{2}}\left(a e^{-iwt} - a^\dagger e^{iwt}\right) \end{aligned}}$$

2)

$$D(t_1 - t_2) = \langle 0 | x(t_1)\, x(t_2) | 0 \rangle$$

$$= \frac{1}{2w} \langle 0 | \left(a e^{-iwt_1} + \cancel{a^\dagger e^{iwt_1}}\right)\left(\cancel{a e^{-iwt_2}} + a^\dagger e^{iwt_2}\right) | 0 \rangle$$

$$= \frac{1}{2w} e^{iw(t_2 - t_1)} \langle 0 | a\, a^\dagger | 0 \rangle$$

$$\boxed{D(t_1 - t_2) = \frac{e^{-iw(t_1 - t_2)}}{2w}}$$

$$(\partial_t^2 + w^2)\, D(t) = (\partial_t^2 + w^2)\, \frac{e^{-iwt}}{2w}$$

$$= \left(\cancel{(-iw)^2 + w^2}^{\,0}\right) \frac{e^{-iwt}}{2w} = 0 \checkmark$$

(3)

If $t_1 \geq t_2$: $\quad D_F(t_1 - t_2) = \langle 0 | T\, x(t_1)\, x(t_2) | 0 \rangle$

$$= \langle 0 | x(t_1)\, x(t_2) | 0 \rangle \Rightarrow D_F(t_1 - t_2) = \frac{e^{-iw(t_1 - t_2)}}{2w}$$

If $t_2 \geq t_1$: $\quad D_F(t_1 - t_2) = \langle 0 | x(t_2)\, x(t_1) | 0 \rangle = D_F(t_2 - t_1)$

$$= \frac{e^{-iw(t_2 - t_1)}}{2w}$$

Combining these:

$$\boxed{D_F(t_1 - t_2) = \frac{e^{-iw|t_1 - t_2|}}{2w}}$$

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(4) (a) When $t_1 \neq t_2$, we know this vanishes from (2),

so the case of interest is $t_1 = t_2$.

At $t_1 = t_2 + \epsilon$, $\epsilon > 0$

$$\partial_{t_1} \langle 0 | T\, x(t_1)\, x(t_2) | 0 \rangle = \frac{1}{2\omega} \partial_{t_1} e^{-i\omega(t_1 - t_2)} = \frac{-i}{2} + O(\epsilon)$$

At $\epsilon < 0$

$$\partial_{t_1} \langle 0 | T\, x(t_1)\, x(t_2) | 0 \rangle = \frac{1}{2\omega} \partial_{t_1} e^{+i\omega(t_1 - t_2)} = +\frac{i}{2} + O(\epsilon).$$

By the definition of the derivative

$$O(\epsilon) + -i = \partial_{t_1} \langle 0 | T\, x(t_2+\epsilon)\, x(t_2) | 0 \rangle - \partial_{t_1} \langle 0 | T\, x(t_2 - \epsilon)\, x(t_2) | 0 \rangle$$

$$= \int_{t_2-\epsilon}^{t_2+\epsilon} dt_1\, \partial_{t_1}^2 \langle 0 | T\, x(t_1)\, x(t_2) | 0 \rangle$$

For all $\epsilon > 0$.

Taking $\lim_{\epsilon \to 0}$ we get

$$\partial_{t_1}^2 \langle 0 | T\, x(t_1)\, x(t_2) | 0 \rangle = -i\, \delta(x_1 - x_2) + \text{finite}.$$

Using (2) we can find the finite contribution, so

$$\boxed{\left(\partial_{t_1}^2 + \omega^2\right) \langle 0 | T\, x(t_1)\, x(t_2) | 0 \rangle = -i\, \delta(t_1 - t_2)}$$

#4

Solution Set #5

(4) (b) The Heisenberg EOMs are

$$\dot{x} = -i[x, H] = -i[x, \tfrac{p^2}{2}] = -\tfrac{i}{2}([x,p]p + p[x,p]) = p$$

$$\ddot{x} = \dot{p} = -i[p, H] = -i[p, \tfrac{1}{2}w^2x^2] = -w^2x$$

$$\partial_\epsilon \langle 0|T\,x(t_2+\epsilon)\,x(t_2)|0\rangle = \partial_\epsilon \langle 0|x(t_2+\epsilon)\,x(t_2)|0\rangle$$

$$= \langle 0|\dot{x}(t_2+\epsilon)\,x(t_2)|0\rangle = \langle 0|p(t_2+\epsilon)\,x(t_2)|0\rangle$$

$$\partial_\epsilon \langle 0|T\,x(t_2-\epsilon)\,x(t_2)|0\rangle = \partial_\epsilon \langle 0|x(t_2)\,x(t_2-\epsilon)|0\rangle$$

$$= -\langle 0|x(t_2)\,p(t_2+\epsilon)|0\rangle$$

$$\int_{-\epsilon}^{\epsilon} \partial^2_{t_1} \langle 0|T x(t_1)\,x(t_2)|0\rangle\, dt_1 = \partial_\epsilon \langle 0|T\,x(t_2+\epsilon)\,x(t_2)|0\rangle - \partial_\epsilon \langle 0|T\,x(t_2-\epsilon)\,x(t_2)|0\rangle$$

$$= \langle 0|[p(t_2+\epsilon), x(t_2)]|0\rangle = -i\,\delta(\epsilon) = -i\,\delta(t_2-t_1)$$

Again, using (2) to see that finite terms vanish

$$\boxed{(\partial^2_{t_1} + w^2)\langle 0|T\,x(t_1)\,x(t_2)|0\rangle = -i\,\delta(t_1-t_2)}$$

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(5) It suffices to consider $t_1 > t_2 > t_3 > t_4$

(because t_i's can always be chosen this way and the answer is invariant under permutations of t_i's)

$$G_4 = \langle 0| T x(t_1)\, x(t_2)\, x(t_3)\, x(t_4) |0\rangle = \langle 0| x(t_1)\, x(t_2)\, x(t_3)\, x(t_4) |0\rangle$$

$$= \frac{1}{4\omega^2} \langle 0| a e^{-i\omega t_1} (a e^{-i\omega t_2} + a^\dagger e^{i\omega t_2})(a e^{-i\omega t_3} + a^\dagger e^{i\omega t_3}) a^\dagger e^{i\omega t_4} |0\rangle$$

$$= \frac{1}{4\omega^2} e^{-i\omega(t_1 + t_2 - t_3 - t_4)} \langle 0| a a a^\dagger a^\dagger |0\rangle$$

$$+ \frac{1}{4\omega^2} e^{-i\omega(t_1 - t_2 + t_3 - t_4)} \langle 0| a a^\dagger a a^\dagger |0\rangle$$

Now to evaluate the matrix elements

$$\langle 0| a a a^\dagger a^\dagger |0\rangle = \langle 0| a [a, a^\dagger a^\dagger] |0\rangle = 2 \langle 0| a a^\dagger |0\rangle = 2$$

$$\langle 0| a a^\dagger a a^\dagger |0\rangle = \langle 0| a a^\dagger [a, a^\dagger] |0\rangle = \langle 0| a a^\dagger |0\rangle = 1$$

$$G_4 = \frac{1}{4\omega^2} \left(2 e^{-i\omega(t_1 + t_2 - t_3 - t_4)} + e^{-i\omega(t_1 - t_2 + t_3 - t_4)} \right)$$

$$D_F(t_1 - t_2) D_F(t_3 - t_4) + D_F(t_1 - t_3) D_F(t_2 - t_4) + D_F(t_1 - t_4) D_F(t_2 - t_3)$$

$$= \frac{1}{4\omega^2} \left(e^{-i\omega(t_1 - t_2 + t_3 - t_4)} + e^{-i\omega(t_1 - t_3 + t_2 - t_4)} + e^{-i\omega(t_1 - t_4 + t_2 - t_3)} \right)$$

$$= G_4$$

$\boxed{\text{So these agree!}}$

2.

a) $a_I(t) = \exp[iE(a^\dagger a)t]\, a\, \exp[-iE(a^\dagger a)t]$

Now $$e^{+iA} B e^{-iA} = B + i[A,B] + \frac{(i)^2}{2!}[A[A,B]] \ldots\ldots$$

So

$$a_I(t) = a + iEt\,(a^\dagger a \overset{\overset{0 \text{ since } a^2=0.}{\shortparallel}}{a} - a a^\dagger a) + \ldots$$

$$= a + iEt\left(-a(-aa^\dagger + 1)\right) + \ldots$$

$$= a - iEt\, a + \ldots$$

$$= a + (-iEt)\,a + \frac{(-iEt)^2}{2!} a.$$

$$= e^{-iEt} a.$$

$$a^\dagger_I(t) = (a_I(t))^\dagger = e^{+iEt} a^\dagger.$$

So $$V_I(t) = V_0\left(a_I(t)\, e^{i\omega t} + a^\dagger_I(t)\, e^{-i\omega t}\right)$$

$$= V_0\left(a\, e^{-i(E-\omega)t} + a^\dagger e^{i(E-\omega)t}\right)$$

b) $${}_I\langle 0(0)|\, T a_I(t_1)\, a^\dagger_I(t_2)\, |0(0)\rangle_I = \theta(t_1-t_2)\, {}_I\langle 0(0)|\, a a^\dagger |0(0)\rangle_I\, e^{-iE(t_1-t_2)}$$

$$+\ \theta(t_2-t_1)\, {}_I\langle 0(0)|\, a^\dagger a\, |0(0)\rangle_I\, e^{iE(t_1-t_2)}, \quad |0(0)\rangle_I = |0\rangle.$$

and $\langle 0|a^\dagger a|0\rangle = 0$ and $\langle 0|\, a a^\dagger |0\rangle = 1$.

$$= \boxed{\theta(t_1-t_2)\, e^{-iE(t_1-t_2)}}$$

c) $\langle 1(t)_I | 0(0)\rangle_I = e^{-iEt} \langle 0 | a\, T\exp\left[-i\int_0^T V_I(t')dt'\right] | 0\rangle$

$$= e^{-iEt} \langle 0| a \left(1 - i\int_0^t V_I(t')dt' + (-i)^2 \int_0^t dt' \int_0^{t'} dt'' V_I(t')V_I(t'') + \ldots\right)|0\rangle$$

To first order in V_0

we have

$$= e^{-iEt}\left(\overset{0}{\overset{\shortparallel}{\langle 0|a|0\rangle}} - iV_0\left(\int_0^t dt'\, e^{-i(E+\omega)t'} \overset{0}{\overset{\shortparallel}{\langle 0|a^2|0\rangle}} + \int_0^t dt'\, e^{i(E-\omega)t'} \langle 0|a a^\dagger|0\rangle\right)\right)$$

$$= e^{-iEt} \frac{-iV_0}{i(E-\omega)}\left(e^{i(E-\omega)t} - 1\right) = e^{-iEt}\frac{V_0}{(E-\omega)}\left(e^{i(E-\omega)t} - 1\right)$$

D)

$$\langle 0(t)|_I |0(0)\rangle_I = \overset{1}{\overset{\shortparallel}{\langle 0|1|0\rangle}} - i\int_0^t \overset{0}{\overset{\shortparallel}{\langle 0|V_I|0\rangle}}$$

$$-V_0^2 \int_0^t dt' \int_0^{t'} dt'' \langle 0|a a^\dagger|0\rangle e^{-i(E-\omega)t'} e^{i(E-\omega)t''}$$

Now $\displaystyle\int_0^t dt' \int_0^{t'} dt''\, e^{-i(E-\omega)t'} e^{i(E-\omega)t''} = \frac{1}{i(E-\omega)}\int_0^t dt'\, e^{-i(E-\omega)t'}\left(e^{i(E-\omega)t'} - 1\right)$

$$= \frac{1 - e^{-i(E-\omega)t}}{(E-\omega)^2} + \frac{t}{i(E-\omega)}$$

So

$$\langle 0(t)|0\rangle = 1 - \frac{V_0^2}{(E-\omega)^2}\left(1-e^{-i(E-\omega)t}\right) - \frac{V_0^2 t}{2i(E-\omega)} = C_0(t).$$

So

$$|C_0|^2 = 1 - \frac{V_0^2}{2(E-\omega)^2}\left(2 - e^{-i(E-\omega)t} - e^{i(E-\omega)t}\right) + O(V_0^4)$$

$$|C_1|^2 = \frac{V_0^2}{(E-\omega)^2}\left(2 - e^{i(E-\omega)t} - e^{-i(E-\omega)t}\right).$$

So $|C_0|^2 + |C_1|^2 = 1$. ✓

e). Now

$$i\frac{d}{dt}|\psi_I(t)\rangle = V_I(t)\,|\psi_I(t)\rangle$$

So $$i\frac{d}{dt}\begin{pmatrix} C_0^I(t) \\ C_1^I(t) \end{pmatrix} = \begin{bmatrix} 0 & V_0 e^{-i(E-\omega)t} \\ V_0 e^{+i(E-\omega)t} & 0 \end{bmatrix}\begin{bmatrix} C_0^I(t) \\ C_1^I(t) \end{bmatrix}$$

So $$i\dot{C}_0^I(t) = V_0 e^{-i(E-\omega)t} C_1^I(t).$$

$$i\dot{C}_1^I(t) = V_0 e^{i(E-\omega)t} C_0^I(t).$$

So $$i\ddot{C}_0^I(t) = -i V_0 (E-\omega) e^{-i(E-\omega)t} C_1^I(t) + V_0 e^{-i(E-\omega)t}(-i)\,i\,\dot{C}_1^I$$

$$= (E-\omega)\dot{C}_0^I(t). - i V_0^2 C_0^I(t).$$

So $$\ddot{C}_0^I(t) + V_0^2 C_0^I(t) + (E-\omega)\,i\,\dot{C}_0^I(t) = 0.$$

Let $\Delta = E - \omega$.

$$\ddot{C}_0 + V_0^2 C_0 + \Delta i \dot{C}_0 = 0.$$

Linear 2nd order differential equation

so

$C_0 \sim e^{imt}$: Plugging in gives

$$(m^2 - V_0^2 - \Delta m) C_0 = 0. \qquad \text{so } m^2 - V_0^2 - \Delta m = 0$$

$$m = \frac{\Delta \pm \sqrt{\Delta^2 + 4V_0^2}}{2} = \frac{\Delta}{2} \pm \sqrt{\left(\frac{\Delta}{2}\right)^2 + V_0^2}$$

so

$$C_0(t) = A e^{i\frac{\Delta}{2}t} e^{it\sqrt{\left(\frac{\Delta}{2}\right)^2 + V_0^2}} + B e^{\frac{i\Delta t}{2}} e^{-it\sqrt{\left(\frac{\Delta}{2}\right)^2 + V_0^2}}$$

Let $g = \sqrt{\left(\frac{\Delta}{2}\right)^2 + V_0^2}$

$\cancel{C_1(t) = -i V_0 e^{i\Delta}}$

$$C_1(t) = \frac{i}{V_0} e^{i\Delta t} \dot{C}_0$$

$$= \frac{i}{V_0} e^{\frac{i3\Delta t}{2}} \left(A\left(i\left(\frac{\Delta}{2} + g\right)\right) e^{igt} + B\left(i\left(\frac{\Delta}{2} - g\right)\right) e^{-itg} \right).$$

So we want.

$$C_0(0) = 1 = A + B.$$

and

$$C_1(0) = 0 = A\left(\frac{\Delta}{2}+g\right) + B\left(\frac{\Delta}{2}-g\right).$$

so
$$A = 1 - B.$$

so $\left(\frac{\Delta}{2}+g\right) - 2gB = 0.$

$$B = \frac{1}{2}\left(\frac{\Delta}{2g} + 1\right)$$

$$A = \frac{1}{2}\left(1 - \frac{\Delta}{2g}\right).$$

so

$$C_0(t) = e^{\frac{i\Delta t}{2}}\left(\cos\sqrt{\left(\frac{\Delta}{2}\right)^2 + V_0^2}\,t - i\frac{\Delta}{2g}\sin\left(\sqrt{\left(\frac{\Delta}{2}\right)^2+V_0^2}\,t\right)\right)$$

$$C_1(t) = \frac{e^{\frac{i3\Delta t}{2}}}{2gV_0}\left(\left(g-\frac{\Delta}{2}\right)\left(\frac{\Delta}{2}+g\right)e^{+itg} - \left(g^2-\frac{\Delta^2}{4}\right)e^{-igt}\right)$$

$$= \frac{e^{\frac{i3\Delta t}{2}}}{2gV_0}\left(g^2-\frac{\Delta^2}{4}\right)\left(e^{itg} - e^{-itg}\right)$$

$$= \frac{i}{V_0 g}\left(g^2-\frac{\Delta^2}{4}\right)e^{\frac{i3\Delta t}{2}}\sin gt = \frac{e^{\frac{i3\Delta t}{2}}\, i}{V_0\sqrt{\frac{\Delta^2}{4}+V_0^2}}\left(\frac{\Delta^2}{4}+V_0^2-\frac{\Delta^2}{4}\right)\sin\left(\sqrt{\frac{\Delta^2}{4}+V_0^2}\,t\right)$$

$$= e^{\frac{i3\Delta t}{2}}\frac{V_0}{\sqrt{V_0^2+\frac{\Delta^2}{4}}}\sin\sqrt{\frac{\Delta^2}{4}+V_0^2}\,t.$$

So

$$|C_0|^2 = \cos^2\left(\sqrt{\left(\tfrac{\Delta}{2}\right)^2 + V_0^2}\, t\right) + \frac{\Delta^2}{4g^2} \sin^2 \sqrt{\left(\tfrac{\Delta}{2}\right)^2 + V_0^2}\, t.$$

$$= \left(-1 + \frac{\Delta^2}{4g^2}\right) \sin^2 \sqrt{\left(\tfrac{\Delta}{2}\right)^2 + V_0^2}\, t \; + 1 =$$

$$= 1 - \frac{4V_0^2}{\Delta^2 + 4V^2} \sin^2 \sqrt{\left(\tfrac{\Delta}{2}\right)^2 + V_0^2}\, t.$$

$$|C_1|^2 = \frac{4V_0^2}{\Delta^2 + 4V_0^2} \sin^2 \sqrt{\left(\tfrac{\Delta}{2}\right)^2 + V_0^2}\, t.$$

with $|C_0|^2 + |C_1|^2 = 1$.

In limit that $V_0 \ll \Delta$.

$$|C_1|^2 \approx \frac{4V_0^2}{\Delta^2}\left(1 - \frac{4V_0^2}{\Delta^2} \cdots\right)\left(\sin^2 \frac{\Delta t}{2}\left(1 + \frac{2V_0^2}{\Delta^2}\right)\right)$$

$$\approx \frac{4V_0^2}{\Delta^2} \sin^2 \frac{\Delta t}{2} + O(V_0^4).$$

From before we had.

$$|C_1|^2 = \frac{V_0^2}{\Delta^2}\, 2\left(1 - \cos \Delta t\right) = \frac{4V_0^2}{\Delta^2} \sin^2 \frac{\Delta t}{2} \quad \checkmark$$

Checks out!