HW #5, due Oct 8

1. Wicks' theorm. Consider a harmonic oscillator, (set $m = 1, \hbar = 1$ for simplicity)

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2. \tag{1}$$

(1) Expand the Heisenberg operator $x(t)$ in terms of the creation and annihilation operators.

(2) Calculate $D(t_1 - t_2) = \langle 0|x(t_1)x(t_2)|0\rangle$. Show that $(\partial_t^2 + \omega^2)D(t) = 0$.

(3) Calculate $D_F(t_1 - t_2) = \langle 0|Tx(t_1)x(t_2)|0\rangle$.

(4) Study $(\partial_{t_1}^2 + \omega^2)\langle 0|Tx(t_1)x(t_2)|0\rangle$. (a) Use the explicit form which you obtained above. (b) Take the second time derivative of the correlation function using the definition of the time-ordered product, Heisenberg equation of motion of $x(t)$, and *equal time* commutation relation $[x(t), p(t)] = i$.

(5) Calculate $\langle 0|Tx(t_1)x(t_2)x(t_3)x(t_4)|0\rangle$ explicitly using creation and annihilation operators. Compare it with

$$D_F(t_1-t_2)D_F(t_3-t_4)+D_F(t_1-t_3)D_F(t_2-t_4)+D_F(t_1-t_4)D_F(t_2-t_3). \tag{2}$$

2. Time-dependent perturbation theory. Consider a Hamiltonian

$$H = Ea^\dagger a + V(t), \tag{3}$$

where a satisfies the anti-commutation relation $\{a, a^\dagger\} = 1$, and the time-dependent potential is given by

$$V(t) = V_0(ae^{i\omega t} + a^\dagger e^{-i\omega t}). \tag{4}$$

We treat the constant V_0 as a small parameter in the perturbative expansion. The Hilbert state consists of two states, the vacuum $a|0\rangle = 0$, and an excited state $|1\rangle = a^\dagger|0\rangle$.

(1) Go to the interaction picture. Write down the operators $a_I(t)$, $a_I^\dagger(t)$ and $V_I(t)$.

(2) Obtain the "Feynman propagator," $\langle 0(0)|_I T a_I(t_1) a_I^\dagger(t_2)|0(0)\rangle_I$.

(3) At $t = 0$, the system was in the "vacuum" state, $|0\rangle$. Use time-dependent perturbation theory at the first order to obtain $c_1(t) = \langle 1(t)|0(0)\rangle$.

(4) Use time-dependent perturbation theory at the second order to obtain $c_0(t) = \langle 0(t)|0(0)\rangle$. Verify the unitarity relation $|c_0(t)|^2 + |c_1(t)|^2 = 1$ within the order of perturbation which we considered. (Hint: $|\Im m(c_0(t))|^2$ is higher order in perturbation theory ($O(V_0^4)$) and hence should be dropped.)

(5) (**optional**) This system can be solved *exactly*. Calculate $|c_1(t)|^2$ and compare it with the perturbative results you obtained above. (It is called Rabi's formula.)