Solution Set #4

Problem I: Notice that P, C commute with c-numbers. I'll also assume $P^2 = C^2 = 1$, $P = P^\dagger$, $C = C^\dagger$.

(1) Plugging (3) into (4) at $t=0$

$$LHS = P\psi(\vec{x},0)P = P\int d\tilde{p} \sum_s \left(a(p,s)u(p,s)e^{-i\vec{p}\cdot\vec{x}} + b^\dagger(p,s)v(p,s)e^{i\vec{p}\cdot\vec{x}}\right)P$$

$$= \boxed{\int d\tilde{p} \sum_s \left(Pa(p,s)P\,u(p,s)e^{-i\vec{p}\cdot\vec{x}} + Pb^\dagger(p,s)P\,v(p,s)e^{i\vec{p}\cdot\vec{x}}\right)}$$

$$= RHS = \gamma^0\psi(-\vec{x},0) = \gamma^0\int d\tilde{p}\sum_s\left(a(p,s)u(p,s)e^{i\vec{p}\cdot\vec{x}} + b^\dagger(p,s)v(p,s)e^{-i\vec{p}\cdot\vec{x}}\right)$$

$$\overset{\text{use}(1)}{=} \int d\tilde{p}\sum_s\left(a(p,s)\,\overset{\text{negate space part only}}{u(-p,s)}\,e^{i\vec{p}\cdot\vec{x}} - b^\dagger(p,s)v(-p,s)e^{-i\vec{p}\cdot\vec{x}}\right)$$

$$\overset{\substack{(p\to -p)\\ d\tilde{p}\to d\tilde{p}}}{=} \boxed{\int d\tilde{p}\sum_s\left(a(-p,s)u(p,s)e^{-i\vec{p}\cdot\vec{x}} - b^\dagger(p,s)v(-p,s)e^{i\vec{p}\cdot\vec{x}}\right)}$$

Now I can match the coefficients of the boxed equations above, as they hold for all $\vec{x}$ and $e^{-i\vec{p}\cdot\vec{x}}$ are lin. indep (except, of course $e^{-i\vec{p}\cdot\vec{x}}$ is $e^{i\vec{p}\cdot\vec{x}}$ when $\vec{p}\to-\vec{p}$, but $u(p,s)$ and $v(p,s)$ are lin indep.). Alternately I could Fourier transform to justify matching these coefficients.

Matching:

$$\boxed{Pa(\vec{p},s)P = a(-\vec{p},s)} \quad \text{and} \quad Pb^\dagger(\vec{p},s)P = -b^\dagger(-\vec{p},s),$$ taking the conj transpose, $P=P^\dagger$

So $\boxed{Pb(\vec{p},s)P = -b(-\vec{p},s)}$

(2) Plugging (3) into (5) at $t=0$

$$LHS = C\psi(\vec{x},0)C = \int d\tilde{p}\sum_s\left(Ca(p,s)C\,u(p,s)e^{-i\vec{p}\cdot\vec{x}} + Cb^\dagger(p,s)C\,v(p,s)e^{i\vec{p}\cdot\vec{x}}\right)$$

$$= RHS = i\gamma^0\gamma^2\bar{\psi}^T(\vec{x},0) = i\gamma^0\gamma^2(\psi^\dagger\gamma^0)^T$$

$$= i\gamma^0\gamma^2\gamma^0(\psi^\dagger)^T = \int d\tilde{p}\sum_s\left(a^\dagger(p,s)\gamma^0\gamma^2\gamma^0(u^\dagger)^T_{(p,s)}e^{+i\vec{p}\cdot\vec{x}} + b(p,s)\gamma^0\gamma^2\gamma^0(v^\dagger)^T_{(p,s)}e^{-i\vec{p}\cdot\vec{x}}\right)$$

$$= \int d\tilde{p}\sum_s\left(a^\dagger(p,s)\gamma^0\gamma^2\bar{u}^T(p,s)e^{+i\vec{p}\cdot\vec{x}} + b(p,s)\gamma^0\gamma^2\bar{v}^T(p,s)e^{-i\vec{p}\cdot\vec{x}}\right)$$

$$= \int d\tilde{p}\sum_s\left(a^\dagger(p,s)v(p,s)e^{+i\vec{p}\cdot\vec{x}} + b(p,s)u(p,s)e^{-i\vec{p}\cdot\vec{x}}\right)$$

Matching: $Ca(p,s)C = b(p,s)$ and $Cb^\dagger(p,s)C = a^\dagger(p,s) \Rightarrow Cb(p,s)C = a(p,s)$

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(3) I'll use the identity (from, e.g. Jackson)

$$Y^{\ell}_{m}(-\vec{p}) = (-1)^{\ell}\, Y^{\ell}_{m}(\vec{p})$$

and I'll use $P|0\rangle = |0\rangle$, parity invariance of the vacuum.

$$P|\ell, m; 0, 0\rangle = \int d^3\vec{p}\,[P a^{\dagger}(\vec{p},+) P P b^{\dagger}(-\vec{p},-) P - P a^{\dagger}(\vec{p},-) P P b^{\dagger}(-\vec{p},+)] \times Y^{\ell}_{m}(\vec{p})\, f(|\vec{p}|)\, P|0\rangle$$

where I've inserted $PP = 1$ and used $P Y^{\ell}_{m}(\vec{p}) P = Y^{\ell}_{m}(\vec{p})$ (c-number)

Now I'll use the results of part (1):

$$P|\ell, m; 0, 0\rangle = \int d^3\vec{p}\,[(a^{\dagger}(-\vec{p},+))(-b^{\dagger}(\vec{p},-)) - (a^{\dagger}(-\vec{p},-))(-b^{\dagger}(\vec{p},+))] \times Y^{\ell}_{m}(\vec{p})\, f(|\vec{p}|)\,|0\rangle$$

$(\vec{p} \to -\vec{p})$

$$= -\int d^3\vec{p}\,[a^{\dagger}(\vec{p},+) b^{\dagger}(-\vec{p},-) - a^{\dagger}(\vec{p},-) b^{\dagger}(-\vec{p},+)]\, Y^{\ell}_{m}(-\vec{p})\, f(|-\vec{p}|)|0\rangle$$

$$= -(-1)^{\ell}\int d^3\vec{p}\,[a^{\dagger}(\vec{p},+) b^{\dagger}(-\vec{p},-) - a^{\dagger}(\vec{p},-) b^{\dagger}(-\vec{p},+)]\, Y^{\ell}_{m}(\vec{p})\, f(|\vec{p}|)|0\rangle$$

$$= \boxed{-(-1)^{\ell}\,|\ell, m; 0, 0\rangle}$$

(4) use $C|0\rangle = |0\rangle$

$$C|\ell, m; 0, 0\rangle = \int d^3\vec{p}\,[C a^{\dagger}(\vec{p},+) C C b^{\dagger}(-\vec{p},-) C - C a^{\dagger}(\vec{p},-) C C b^{\dagger}(-\vec{p},+)]\, Y^{\ell}_{m}(\vec{p})\, f(|\vec{p}|)|0\rangle$$

$$= \int d^3\vec{p}\,[b^{\dagger}(\vec{p},+) a^{\dagger}(-\vec{p},-) - b^{\dagger}(\vec{p},-) a^{\dagger}(-\vec{p},+)]\, Y^{\ell}_{m}(\vec{p})\, f(|\vec{p}|)|0\rangle$$

(use $a^{\dagger}b^{\dagger} = -b^{\dagger}a^{\dagger}$)

$$= -\int d^3\vec{p}\,[a^{\dagger}(-\vec{p},-) b^{\dagger}(\vec{p},+) - a^{\dagger}(-\vec{p},+) b^{\dagger}(\vec{p},-)]\, Y^{\ell}_{m}(\vec{p})\, f(|\vec{p}|)|0\rangle$$

$$= \int d^3\vec{p}\,[a^{\dagger}(-\vec{p},+) b^{\dagger}(\vec{p},-) - a^{\dagger}(-\vec{p},-) b^{\dagger}(\vec{p},+)]\, Y^{\ell}_{m}(\vec{p})\, f(|\vec{p}|)|0\rangle$$

$(\vec{p} \to -\vec{p})$

$$= \int d^3\vec{p}\,[a^{\dagger}(\vec{p},+) b^{\dagger}(-\vec{p},-) - a^{\dagger}(\vec{p},-) b^{\dagger}(-\vec{p},+)]\, Y^{\ell}_{m}(-\vec{p})\, f(|\vec{p}|)|0\rangle$$

$$= (-1)^{\ell}\int d^3\vec{p}\,[a^{\dagger}(\vec{p},+) b^{\dagger}(-\vec{p},-) - a^{\dagger}(\vec{p},-) b^{\dagger}(-\vec{p},+)]\, Y^{\ell}_{m}(\vec{p})\, f(|\vec{p}|)|0\rangle$$

$$= \boxed{(-1)^{\ell}\,|\ell, m; 0, 0\rangle}$$

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(5) $P|l,m;1,1\rangle = \int d^3\vec{p}\,[Pa^\dagger(\vec{p},+)PPb^\dagger(-\vec{p},+)P]\,Y^l_m(\hat{p})f(|\vec{p}|)\,P|0\rangle$

$= -\int d^3\vec{p}\; a^\dagger(-\vec{p},+)\,b^\dagger(\vec{p},+)\,Y^l_m(\hat{p})f(|\vec{p}|)\,|0\rangle$

$(\vec{p}\to-\vec{p})\quad = -\int d^3\vec{p}\; a^\dagger(\vec{p},+)\,b^\dagger(-\vec{p},+)\,Y^l_m(-\hat{p})\,f(|\vec{p}|)\,|0\rangle$

$= -(-1)^l\;|l,m;1,1\rangle$

(6) $C|l,m;1,1\rangle = \int d^3\vec{p}\,[Ca^\dagger(\vec{p},+)CC\,b^\dagger(-\vec{p},+)C]\,Y^l_m(\hat{p})\,f(|\vec{p}|)\,C|0\rangle$

$= \int d^3\vec{p}\,[\,b^\dagger(\vec{p},+)\,a^\dagger(-\vec{p},+)]\,Y^l_m(\hat{p})f(|\vec{p}|)\,|0\rangle$

$(a^\dagger b^\dagger = -b^\dagger a^\dagger)\quad = -\int d^3\vec{p}\;[a^\dagger(-\vec{p},+)\,b^\dagger(\vec{p},+)]\,Y^l_m(\hat{p})\,f(|\vec{p}|)\,|0\rangle$

$(\vec{p}\to-\vec{p})\quad = -\int d^3\vec{p}\;[a^\dagger(\vec{p},+)\,b^\dagger(-\vec{p},+)]\,(-1)^l\,Y^l_m(\hat{p})f(|\vec{p}|)\,|0\rangle$

$= \boxed{-(-1)^l\,|l,m;1,1\rangle}$

Problem II: Use $P=(-1)^{L+1}$, $C=(-1)^{L+S}$ for fermion pair bound states

(1) $L=0$ so $P=(-1)^{L+1}=-1$ so these are $\boxed{\text{pseudoscalars}}$

(2) I'll guess:

$$\boxed{|K_1\rangle = |K^0\rangle - |\bar{K}^0\rangle\;,\;|K_2\rangle = |K^0\rangle + |\bar{K}^0\rangle}$$

Now I'll check this.

First notice $\quad CP\,|K^0\rangle = (-1)^{L+S}(-1)^{L+1}\,|\bar{K}^0\rangle = -|\bar{K}^0\rangle$

$CP\,|\bar{K}^0\rangle = (-1)^{2L+S+1}\,|K^0\rangle = -|K^0\rangle$

And so

$CP\,|K_1\rangle = CP\,(|K^0\rangle - |\bar{K}^0\rangle) = -|\bar{K}^0\rangle + |K^0\rangle = |K_1\rangle$

$CP\,|K_2\rangle = CP(|K^0\rangle + |\bar{K}^0\rangle) = -|\bar{K}^0\rangle - |K^0\rangle = -|K_2\rangle$

So $|K_1\rangle$ and $|K_2\rangle$ are CP eigenstates with eigenvalues 1 and -1 respectively as desired.

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(3) π^0 is a 2 fermion bound state, so $P = (-1)^{L+1} = -1$, $C = (-1)^{L+S} = 1$

$\overline{\pi^0} = \overline{u\bar{u} - d\bar{d}} = \bar{u}u - \bar{d}d = \pi^0$, so π^0 is it's own antiparticle.

Let $a^+(\vec{p})$ create a π^0 with momentum $\vec{p}$. By above

$$P a^+(\vec{p})|0\rangle = -a^+(-\vec{p})|0\rangle \quad \text{because } P = -1$$

however

$$P a^+(\vec{p})|0\rangle = P a^+(\vec{p}) P P|0\rangle = P a^+(\vec{p}) P|0\rangle$$

And so, taking from the Lecture of Sept 10, 1999

$$P a^+(\vec{p}) P = \eta_p a^+(-\vec{p})$$

we find $\eta_p = -1$, so

$$\boxed{P a^+(\vec{p}) P = -a^+(-\vec{p})}$$

Likewise, as $\overline{\pi^0} = \pi^0$,

$$\boxed{C a^+(\vec{p}) C = a^+(\vec{p})}$$

Now to create a state of $2\pi^0$ with no angular momentum:

$$\int d\hat{p}\; a^+(\vec{p})\, a^+(-\vec{p})\, \cancel{Y_0^0(\hat{p})}^{\text{const}}\, f(|\vec{p}|)|0\rangle$$

$$CP \int d\hat{p}\, a^+(\vec{p})\, a^+(-\vec{p})\, f(|\vec{p}|)|0\rangle = \int d\hat{p}\, (CP\, a^+(\vec{p})\, PC)(CP a^+(-\vec{p})\, PC)\, f(|\vec{p}|)\, CP|0\rangle$$

$$= \int d\hat{p}\, (-a^+(\vec{p}))(-a^+(-\vec{p}))\, f(|\vec{p}|)|0\rangle$$

$$= \int d\hat{p}\; a^+(\vec{p})\, a^+(-\vec{p})\, f(|\vec{p}|)|0\rangle$$

which is just the original state. So this bound state has $CP = 1$.

(4) $CP\; a^+(\vec{p}_1)\, a^+(\vec{p}_2)\, a^+(\vec{p}_3) f|0\rangle = (-1)^3\, a^+(\vec{p}_1)\, a^+(\vec{p}_2)\, a^+(\vec{p}_3) f|0\rangle$

similarly to above, so this state (this is the 0 ang mom state) will have $CP = -1$, like K_2

Therefore $\boxed{K_2 \to \pi^0\pi^0\pi^0 \text{ is allowed}}$.