HW #4, due Oct 1

1. The P and C transformation of fermion pair bound states. It is customary to use Pauli–Dirac representation for γ-matrices when one is dealing with non-relativistic fermions, especially their bound states. You first start with the solutions at rest $p^\mu = \pm m(1,0,0,0)$ which are eigenstates of s_z (spin along the z axis). Then you boost the Lorentz frame to obtain four-momentum along arbitrary directions. In this way, you obtain two positive energy solutions labeled as $u(p,s)$, where $s = \pm 1/2$ is the eigenvalue of s_z *before* the boost. Similarly, you obtain two negative energy solutions $v(p,s)$. You can find explicit expressions for $u(p,s)$ and $v(p,s)$ in many textbooks, *e.g.* the one by Bjorken and Drell. By defining $u(p,s)$ and $v(p,s)$ in the above manner, you can choose your basis such that the following relations hold (far easier than the helicity basis!):

$$\gamma^0 u(\vec{p},s) = u(-\vec{p},s), \qquad \gamma^0 v(\vec{p},s) = -v(-\vec{p},s), \tag{1}$$

$$i\gamma^0\gamma^2 \overset{T}{\bar{u}}(\vec{p},s) = v(\vec{p},s), \qquad i\gamma^0\gamma^2 \overset{T}{\bar{v}}(\vec{p},s) = u(\vec{p},s). \tag{2}$$

We expand the Dirac field in the usual way also with this basis, *i.e.*,

$$\psi(x) = \int d\tilde{p} \sum_s (a(p,s)u(p,s)e^{-ip\cdot x} + b^\dagger(p,s)v(p,s)e^{ip\cdot x}) \tag{3}$$

and define the parity and charge conjugation by

$$P\psi(\vec{x},t)P = \gamma^0\psi(-\vec{x},t) \tag{4}$$

$$C\psi(\vec{x},t)C = i\gamma^0\gamma^2 \overset{T}{\bar{\psi}}(\vec{x},t) \tag{5}$$

(1) Show that the mode operators satisfy the following relations, $Pa(\vec{p},s)P = a(-\vec{p},s)$, $Pb(\vec{p},s)P = -b(-\vec{p},s)$.

(2) Show that the mode operators satisfy the following relations, $Ca(\vec{p},s)C = b(\vec{p},s)$, $Cb(\vec{p},s)C = a(\vec{p},s)$.

(3) Define a state $|L, L_z; S, S_z\rangle$ with $L = l$ and $S = 0$ by

$$\begin{aligned} &|l,m;0,0\rangle \\ &= \int d^3\vec{p}\left[a^\dagger(\vec{p},+1/2)b^\dagger(-\vec{p},-1/2) - a^\dagger(\vec{p},-1/2)b^\dagger(-\vec{p},+1/2)\right] \times \\ &\quad Y^l_m(\hat{\vec{p}})f(|\vec{p}|)|0\rangle \end{aligned} \tag{6}$$

Show that $P|l,m;0,0\rangle = -(-1)^l|l,m;0,0\rangle$. Here, $Y^l_m(\hat{\vec{p}})$ is defined by the direction of $\hat{\vec{p}} = \vec{p}/|\vec{p}|$ while $f(|\vec{p}|)$ is the "radial" part which depends only on the size of the momentum.

(4) Show that $C|l,m;0,0\rangle = (-1)^l|l,m;0,0\rangle$. Since a photon has an odd eigenvalue under C, $l=0$ state can decay into two photons. Examples include a para-positronium ($L=S=0$ bound state of an electron and a positron), π^0 meson and η^0 meson (both $u\bar{u}$ and $d\bar{d}$ bound states in $L=S=0$ channel).

(5) Define a state with $L=l$ and $S=1$ by

$$|l,m;1,1\rangle = \int d^3\vec{p}\left[a^\dagger(\vec{p},+1/2)b^\dagger(-\vec{p},+1/2)\right]Y^l_m(\hat{\vec{p}})f(|\vec{p}|)|0\rangle \qquad (7)$$

Show that $P|l,m;1,1\rangle = -(-1)^l|l,m;1,1\rangle$. This is the same eigenvalue as the $S=0$ case.

(6) Show that $C|l,m;1,1\rangle = -(-1)^l|l,m;1,1\rangle$, which is the opposite eigenvalue from the $S=0$ case. $l=0$ state can hence decay into three photons. Examples include an ortho-positronium ($L=0$, $S=1$) decaying into three photons, and a J/ψ particle ($L=0$, $S=1$ bound state of charm and anti-charm quark) which decays into three "gluons".

Note It is usually summarized as $P=(-1)^{L+1}$ and $C=(-1)^{L+S}$.

2. CP**.** There are neutral Kaons, K^0 and $\overline{K}^0$, distinguished by their "strangeness", 1 and -1, respectively. In the quark model, they are $d\bar{s}$ and $s\bar{d}$ bound states. Assume that CP is a good quantum number.

(1) Both of them are $L=S=0$ bound states. Are they scalars or pseudo-scalars?

(2) Assume $C|K^0\rangle = |\overline{K}^0\rangle$ and vice versa. Write down eigenstates of CP operator as linear combinations of $|K^0\rangle$ and $|\overline{K}^0\rangle$. The $CP=1$ state is called K_1, and $CP=-1$ state K_2.

(3) Another neutral meson, π^0 is a bound state of $u\bar{u}$ and $d\bar{d}$ in $L=S=0$ channel. (Never mind it is actually a $u\bar{u}-d\bar{d}$ combination.) If you have two π^0 with no relative angular momentum, show that it is a state with $CP=1$, using a similar technique as in Problem **1.**, but with creation operators of a Klein-Gordon field. Therefore, K_1 can decay into $2\pi^0$ but K_2 cannot.

(4) Can K_2 decay into $3\pi^0$? (Assume it is kinematically allowed)

Note Experimentally, it was observed that K_2 can actually decay occasionally into $2\pi^0$. This was the only pheomenon known to 1998 which violates CP invariance.