HW#3 SOLUTION SET

1. $\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ so $\gamma^0\gamma^0 + \gamma^0\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2\cdot\mathbb{1} = 2g^{00}$

For $i \neq 0$

$$\{\gamma^0 \gamma^i\} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix} + \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\sigma_i \\ -\sigma_i & 0 \end{bmatrix} = 0 = 2g^{0i} \quad \text{for } i \neq 0.$$

and $\{\gamma^i \gamma^j\} = 2g^{ij}$ from handouts before.

b) The Dirac eq. is given by

$(i\gamma^\mu \partial_\mu - m)\psi = 0$ Let $\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$

make substitution of $\nabla \to \nabla - ieA$.

you get

$$\begin{pmatrix} i\frac{\partial}{\partial t} - m & i\sigma\cdot(\nabla - ieA) \\ -i\sigma\cdot(\nabla - ieA) & -(i\frac{\partial}{\partial t} + m) \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0.$$

$$i\frac{\partial\phi}{\partial t} - m\phi + i\sigma\cdot(\nabla - ieA)\chi = 0.$$

$$i\frac{\partial\chi}{\partial t} + m\chi + i\sigma\cdot(\nabla - ieA)\phi = 0.$$

c) Let $\phi' = e^{imt}\phi \quad \chi' = e^{imt}\chi.$

So $i\frac{\partial\phi'}{\partial t} = i\left(im\phi' + \cancel{i}\, e^{imt}\frac{\partial\phi}{\partial t}\right) = ie^{imt}\frac{\partial\phi}{\partial t} - me^{imt}\phi$

and $i\frac{\partial\chi'}{\partial t} = ie^{imt}\frac{\partial\chi}{\partial t} - me^{imt}\chi.$

So multiplying both equations by e^{imt} you get

$$i\frac{\partial \phi'}{\partial t} + i\sigma\cdot(\nabla - ieA)\chi' = 0.$$

$$i\frac{\partial \chi'}{\partial t} + 2m\chi' + i\sigma\cdot(\nabla - ieA)\phi' = 0$$

D) Now ϕ' and $\chi' \propto e^{-iEt} e^{imt} \approx e^{-i\frac{mv^2}{2}}$ In NR limit.

So $\frac{\partial}{\partial t}\phi' \approx \frac{mv^2}{2}\phi' \approx O(mv^2)$. since ϕ' is $O(1)$.

and $\frac{\partial \chi'}{\partial t} \approx \frac{mv^2}{2}\chi' \approx O(mv^3)$ since χ' is of $O(v)$

So Looking at 2nd equation

$$2m\chi' + i\sigma.(\nabla - ieA)\phi' = 0.$$

So $\chi' = -i\frac{\sigma.(\nabla - ieA)\phi'}{2m} + O(mv^3)$.

e) So plugging into first equation.

$$i\frac{\partial \phi'}{\partial t} = -i\sigma.(\nabla - ieA)\chi' = -\frac{(\sigma.(\nabla - ieA))^2}{2m}\phi' + O(mv^3)$$

$$= \frac{(\sigma.(-i\nabla - eA))^2}{2m}\phi' + O(mv^3).$$

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f) Now

$$(\sigma\cdot(-i\nabla-eA))^2 = \sigma_i\sigma_j\,(-i\nabla-eA)_i\,(-i\nabla-eA)_j$$

Using $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$.

$$= \delta_{ij}(-i\nabla-eA)_j(-i\nabla-eA)_i + i\epsilon_{ijk}\sigma_k(-i\nabla-eA)_i(-i\nabla-eA)_j$$

$$= (-i\nabla-eA)^2 + i\,\sigma\cdot(i\nabla + eA)\times(i\nabla+eA)$$

$$= (-i\nabla-eA)^2 + i\,\sigma\cdot\Big(-\underset{\substack{\parallel\\0}}{\nabla\times\nabla} + e^2\underset{\substack{\parallel\\0}}{A\times A} + ie\,[\nabla\times A + A\times\nabla]\Big).$$

Now

$$(\nabla\times A\phi') = \epsilon_{jkl}(\partial_j(A_k\phi')) = \epsilon_{jkl}((\partial_jA_k)\phi' + A_k\partial_j\phi').$$

$$= (\nabla\times A)\phi' - (A\times\nabla)\phi'$$

So

$$(\nabla\times A + A\times\nabla)\phi' = (\nabla\times A)\phi'$$

but $\nabla\times A = \vec{B}$.

So

$$(\sigma\cdot(-i\nabla-eA))^2 = (-i\nabla-eA)^2 - e\,\sigma\cdot B.$$

So

$$i\frac{\partial\phi'}{\partial t} = \frac{(-i\nabla-eA)^2}{2m} - \frac{e}{2m}\,\sigma\cdot B = \frac{(-i\nabla-eA)^2}{2m} - \frac{ge}{2m}\,\frac{\sigma}{2}\cdot B$$

$$g = 2$$

2. $\vec{J} = \int d^3x :\psi^\dagger\left(-i\vec{x}\times\vec{\nabla} + \frac{\vec{\sigma}}{2}\right)\psi$

a) Now $\vec{P}\, a^\dagger_\pm(p)|0\rangle = \vec{P}|\vec{p},\pm\rangle = \vec{p}\, a^\dagger_\pm(p)|0\rangle$

So $J\cdot\vec{P}\, a^\dagger_\pm(p)|0\rangle = \vec{p}\cdot J\, a^\dagger_\pm(p)|0\rangle$.

From handout.

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3 2E_p}\left(u_s(p)\, a_s(p)\, e^{-ip\cdot x} + v_s(p)\, b^\dagger_s(p)\, e^{+ip\cdot x}\right)$$

So $$-i\vec{x}\times\nabla\psi(x) = \sum_s \frac{d^3p}{(2\pi)^3 2E_p}(x\times\vec{p})\left(v_s(p)\, b^\dagger_s(p)\, e^{+ip\cdot x} - u_s(p)\, a_s(p)\, e^{-ip\cdot x}\right)$$

So

$$p\cdot J|p,\pm\rangle = \sum_{ss'}\int dx^3 \int\frac{dq^3}{(2\pi)^3}\int\frac{dp'^3}{(2\pi)^3}\frac{\vec{p}\cdot}{2E_q\, 2E_{p'}}\Big(:u^\dagger_s(q)\left(\vec{x}\times\vec{p}' + \frac{\vec{\sigma}}{2}\right)v_{s'}(p')\, e^{i(p'+q)\cdot x}\, a^\dagger_s(q)\, b^\dagger_{s'}(p')$$

$$+ v^\dagger_s(q)\left(x\times p' + \frac{\sigma}{2}\right)v_{s'}(p')\, e^{i(p'-q)\cdot x}\, b_s(q)\, b^\dagger_{s'}(p') - u^\dagger_s(q)\left(\vec{x}\times\vec{p}' - \frac{\vec{\sigma}}{2}\right)u_{s'}(p')\, a_s(q)\, a_{s'}(p')\, e^{i(q-p')\cdot x}$$

$$- v^\dagger_s(q)\left(\vec{x}\times\vec{p}' - \frac{\vec{\sigma}}{2}\right)u_{s'}(p')\, e^{-i(q+p')\cdot x}\, b_s(q)\, a_{s'}(p'):\; a^\dagger_\pm(p)|0\rangle$$

$=$ term 1 + term 2 − term 3 − term 4.

Now term 2 and term 4 $=0$ since $:b_s(q)\, b^\dagger_{s'}(p'):\, a^\dagger_\pm(p)|0\rangle$

since $\{b_s(q)\, a^\dagger_\pm(p)\} = 0$ and $b_s(q)|0\rangle = 0$.

$\{b_s(q)\, a_{s'}(p')\} = 0$.

Now look at the 3RD term.

$$a^+_s(q)\, a_{s'}(p')\, a^+_\pm(p)\,|0\rangle = a^+_s(q)\left(\{a_{s'}(p'), a^+_\pm(p)\} - a^+_\pm(p)\, a_{s'}(p')\right)|0\rangle$$

and $\{a_{s'}(p')\, a^+_\pm(p)\} = 2E_p\, \delta^3(p-p')\, \delta_{\pm,s'}\, (2\pi)^3$

so the forth term gives

$$= -\sum_{s s'} \int dx^3 \int \frac{dq^3}{(2\pi)^3} \int \frac{dp'^3}{(2\pi)^3} \frac{(p\cdot(x\times p')}{2E_q\, 2E_{p'}}\, U^+_s(q)\, U_{s'}(p')\, 2E_p\, \delta^3(p-p')\, \delta_{\pm s'}(2\pi)^3 e^{i(q-p')\cdot x}$$

$$- p\cdot U^+_s(q)\, \frac{\sigma}{2}\, U_{s'}(p')\, e^{i(q-p')\cdot x} (2E_p (2\pi)^3 \delta(q-p')\delta_{\pm s'}\Big)\, a^+_{s'}(q)\,|0\rangle.$$

Integrate over p' and sum over s' gives

$$-\sum_s \int dx^3 \int \frac{dq^3}{(2\pi)^3}\left(\frac{p\cdot(x\times p)}{2E_q}\, U^+_s(q)\, U_\pm(p)\, e^{i(q-p)\cdot x} - p\cdot U^+_s(q)\, \frac{\sigma}{2}\, U_\pm(p)\, e^{i(q-p)\cdot x}\right) a^+_{s'}(q)\,|0\rangle$$

Now $p\cdot(x\times p) = 0$; integrating over x $\int dx^3 e^{i(q-p)\cdot x} = (2\pi)^3 \delta^3(q-p)$.
so one gets.

$$= \sum_s \int dq^3\, \delta^3(q-p)\, \frac{U^+_s(q)}{2E_q}\, \frac{p\cdot\sigma}{2}\, U_\pm(p)\, a^+_s(q)\,|0\rangle.$$

$$= \sum_s \frac{U^+_s(p)}{2E_p}\, \frac{p\cdot\sigma}{2}\, U_\pm(p)\, a^+_s(p)\,|0\rangle$$

Now from handout

$$p\cdot\sigma\ \chi_{\pm}(p) = \pm|p|\chi_{\pm}\ p \qquad \chi_+^\dagger(p)\chi_+ = \chi_-^\dagger\chi_- = 1$$

$$\chi_-^\dagger\chi_+ = \chi_+^\dagger\chi_- = 0.$$

$$U_+(p) = \begin{pmatrix}\sqrt{E-p}\\ \sqrt{E+p}\end{pmatrix}\chi_+(p) \qquad U_-(p) = \begin{pmatrix}\sqrt{E+p}\\ \sqrt{E-p}\end{pmatrix}\chi_-p. \qquad U_s^\dagger U_{s'} = \delta_{ss'}2E$$

So

$$\frac{\pm|p|}{2}\ \frac{U_s^\dagger(p)\,U_\pm(p)}{2E_p} = \frac{\pm|p|}{2}\delta_{s,\pm}.$$

So this term gives, after summing over s

$$\boxed{\frac{\pm|p|}{2}\,a_\pm^\dagger(p)|0\rangle.}$$ Now I must show that the first term is zero.

$$\sum_{ss'}\int dx^3\int\frac{dq^3}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}\ \frac{p\cdot\left(U_s^\dagger(q)\left(\vec{x}\times p'+\frac{\sigma}{2}\right)V_{s'}(p')\,e^{i(p'+q)\cdot x}\right.}{2E_q\,2E_{p'}}\ a_s^\dagger(q)\,b_{s'}^\dagger(p')\,a_\pm^\dagger(p)|0\rangle$$

Now

$$\int d^3x\ \vec{x}\ e^{i(p'+q)\cdot x} = -i\vec{\nabla}_{p'}\int d^3x\, e^{i(p'+q)\cdot x} = -i\nabla_{p'}\,\delta^3(p'+q)(2\pi)^3.$$

So one has

$$\sum_{s s'} \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3p'}{2E_{p'} 2E_q} \vec{p} \cdot U^\dagger_{s(q)} \left(-i\nabla_{p'} \left(\delta^3(p'+q) \right) \times p' + \frac{\sigma}{2} \delta^3(p'+q) \right) V_{s'}(p')\, a^\dagger_s(q) b^\dagger_{s'}(p') a_{\pm}(p)|0\rangle$$

Now I integrate over d^3p'.

Now

$$\int d^3p' \frac{}{2E_{p'}} \left(\nabla_{p'} \delta^3(p'+q) \times \left(p' V_{s'}(p') b^\dagger_{s'}(p') \right) \right)$$

$$= \text{surface term} - \int d^3p' \, \delta^3(p'+q)\, \epsilon_{ijk}\, p_j \frac{\partial}{\partial p_i} \left(\frac{V_{s'}(p') b^\dagger_{s'}(p')}{2E_{p'}} \right)$$

$$= + \int d^3p' \, \delta^3(p'+q)\, \vec{p}' \times \frac{\partial}{\partial \vec{p}'} \left(\frac{V_{s'}(p') b^\dagger_{s'}(p')}{2E_{p'}} \right).$$

Now $\frac{\partial}{\partial p'_i} \frac{1}{E_{p'}} = \frac{\partial}{\partial p'_i} \frac{1}{\sqrt{p^2+m^2}} = \frac{p_i}{(p^2+m^2)^{3/2}}$

So $\vec{p}' \times \frac{\partial}{\partial \vec{p}'} \frac{1}{E_{p'}} \propto \vec{p}' \times \vec{p}' = 0.$

Similarly

$$\frac{\partial}{\partial p_i} b^\dagger_{s'}(p')|0\rangle \propto p_i\, b^\dagger_s(p)|0\rangle.$$

So $p' \times \frac{\partial}{\partial p'} b^\dagger_{s'}(p')|0\rangle \propto p' \times p' \, b^\dagger_{s'}(p')|0\rangle = 0.$

So only the $p' \times \frac{\partial (v_s(p))}{\partial p'}$ contributes.

So one gets after integrating

$$\sum_{ss'} \int \frac{d^3q}{(2\pi)^3} \frac{\vec{P}\cdot}{(2E_q)^2} \left(U_s^+(q) \left(-i\, \vec{q} \times \frac{\partial}{\partial \vec{q}} v_{s'}(-q) + \frac{\vec{\sigma}}{2} v_{s'}(-q) \right) a_s^+(q)\, b_{s'}^+(-q)\, a_\pm^+(p) |0\rangle \right.$$

Now one can choose $\vec{q}$ to be along a $\hat{z}$ axis and take $\vec{P}$ to lie along the z-y plane (since each operator $a_s^+(q) b_s^+(-q) a_\pm^+(p)$ are linearly independent for each p, q, the coefficients of each must vanish seperately and we choose the co-ordinate system for each p and q.)

Now look at

$$\chi_+(q) = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1+\frac{q_z}{q}}{2}} \\ \frac{q_x + i q_y}{2q\sqrt{\frac{1+\frac{q_z}{q}}{2}}} \end{pmatrix} \qquad \chi_-(q) = \begin{pmatrix} -\frac{(q_x - i q_y)}{2p\sqrt{\frac{1+\frac{q_z}{q}}{2}}} \\ \sqrt{\frac{1+\frac{q_z}{q}}{2}} \end{pmatrix}$$

$$\frac{\partial (\chi_+(q))}{\partial q_y}\Big|_{\vec{q}=(0,0,q)} = \begin{pmatrix} 0 \\ \frac{i}{2q} \end{pmatrix} \qquad \frac{\partial (\chi_+(q))}{\partial q_x}\Big|_{q=(0,0,q)} = \begin{pmatrix} 0 \\ \frac{1}{2q} \end{pmatrix}$$

$$\frac{\partial (\chi_-(q))}{\partial q_y}\Big|_{q=(0,0,q)} = \begin{pmatrix} \frac{i}{2q} \\ 0 \end{pmatrix} \qquad \frac{\partial (\chi_-(q))}{\partial q_x}\Big|_{q=(0,0,q)} = \begin{pmatrix} -\frac{1}{2q} \\ 0 \end{pmatrix}$$

Now $V_+(-q) = \begin{pmatrix} -\sqrt{E-q} \\ \sqrt{E+q} \end{pmatrix} \chi_+(-q)$.

$$V_-(-q) = \begin{pmatrix} \sqrt{E+q} \\ -\sqrt{E-q} \end{pmatrix} \chi_-(-q).$$

Now $\frac{\partial}{\partial \vec{q}} \sqrt{E-q} \propto \vec{q}$

and $\frac{\partial}{\partial \vec{q}} \sqrt{E+q} \propto \vec{q}$

So $q \times \frac{\partial}{\partial \vec{q}} \sqrt{E \pm q} = 0$.

So

$$i q \times \frac{\partial}{\partial q} V_+(-q) = \begin{pmatrix} -\sqrt{E-q} \\ \sqrt{E+q} \end{pmatrix} q \times \frac{\partial}{\partial q} \chi_+(-q) = \begin{pmatrix} -\sqrt{E-q} \\ \sqrt{E+q} \end{pmatrix} \left[\begin{pmatrix} \frac{-i}{2} \\ 0 \end{pmatrix} \hat{x} + \begin{pmatrix} \frac{-1}{2} \\ 0 \end{pmatrix} \hat{y} \right]$$

$$i q \times \frac{\partial}{\partial q} V_-(-q) = \begin{pmatrix} \sqrt{E+q} \\ -\sqrt{E-q} \end{pmatrix} q \times \frac{\partial}{\partial q} \chi_-(-q) = \begin{pmatrix} \sqrt{E+q} \\ -\sqrt{E-q} \end{pmatrix} \left[\begin{pmatrix} 0 \\ -\frac{i}{2} \end{pmatrix} \hat{x} + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \hat{y} \right]$$

So $\vec{p} \cdot \left(-i \vec{q} \times \frac{\partial}{\partial \vec{q}} V_+(-q) + \frac{\vec{\sigma}}{2} V_+(-q) \right)$.

$$= \begin{pmatrix} -\sqrt{E-q} \\ +\sqrt{E+q} \end{pmatrix} \left(\underbrace{p_y \begin{pmatrix} +\frac{i}{2} \\ 0 \end{pmatrix} + \frac{p_y}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{=0} + p_z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$= p_z \begin{pmatrix} -\sqrt{E-q} \\ +\sqrt{E+q} \end{pmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Now $U_+^\dagger(q) = \begin{pmatrix} \sqrt{E-q} \\ \sqrt{E+q} \end{pmatrix} (1, 0).$ $\quad U_-^\dagger(q) = \begin{pmatrix} \sqrt{E+q} \\ \sqrt{E-q} \end{pmatrix} (0 \ 1).$

So $U_+^\dagger(q) \begin{pmatrix} -\sqrt{E-q} \\ +\sqrt{E+q} \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = 0.$

$$U_-^\dagger(q) \begin{pmatrix} -\sqrt{E-q} \\ \sqrt{E+q} \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{E+q} & 0 & \sqrt{E-q} \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{E-q} \\ 0 \\ -\sqrt{E+q} \end{pmatrix} = 0.$$ My God!..!

Now Let's look at the V_- term.

$$p \cdot \left(-i \vec{q} \times \frac{\partial}{\partial \vec{q}} V_-(-q) + \frac{\vec{\sigma}}{2} V_-(-q) \right)$$

$$= \begin{pmatrix} \sqrt{E+q} \\ -\sqrt{E-q} \end{pmatrix} \left(\underbrace{p_y \begin{pmatrix} 0 \\ -\frac{i}{2} \end{pmatrix} + \frac{p_y}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{=0} + p_z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$= \begin{pmatrix} \sqrt{E+q} \\ -\sqrt{E-q} \end{pmatrix} p_z \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

So check.

$$U_+^\dagger(q)\begin{pmatrix}\sqrt{E+q}\\-\sqrt{E-q}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\left(\sqrt{E+q}\;\;0\;\;\sqrt{E+q}\;\;0\right)\begin{pmatrix}\sqrt{E+q}\\0\\-\sqrt{E-q}\\0\end{pmatrix}=0.$$

$$U_-^\dagger(q)\begin{pmatrix}\sqrt{E+q}\\-\sqrt{E-q}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\left(0\;\sqrt{E+q}\;\;0\;\sqrt{E-q}\right)\begin{pmatrix}\sqrt{E+q}\\0\\-\sqrt{E-q}\\0\end{pmatrix}=0.$$

Thus the term vanishes!!!.

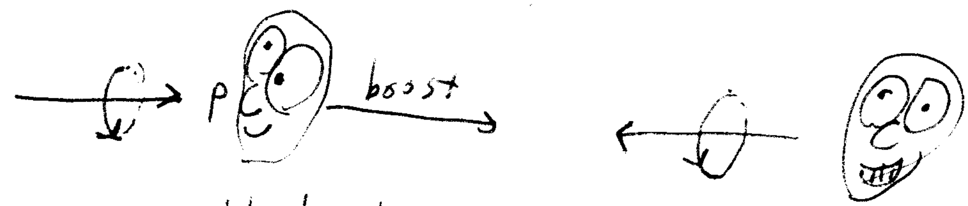
Thus we've proven that

$$J\cdot P\, a_\pm^\dagger|0\rangle = \pm\frac{|P|}{2}\, a_\pm^\dagger|0\rangle.$$

b. Now this part follows analogously to the previous part. however one must remember that $:b_s(q)b_s^\dagger(p'):\ = -b_s^\dagger(p')\,b_s(q)$ Since the $\{b^\dagger(p'), b(p)\}$ anti commute

Everything follows the same after that. so you've just picked up an extra $-$ sign. so

$$J\cdot p\, b_\pm^\dagger|0\rangle = \mp\frac{|p|}{2}\, b_\pm^\dagger|0\rangle.$$

C) 

We see that by being able to catch up with a partice we see that the helicity has changed from + to −. This is the reason why states with nonzero mass must have 2 helicity states.

D)

$$\mathcal{L}_{Dirac} = \bar{\Psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi.$$

For $m = 0$.

$$\mathcal{L}_{Dirac} = i\bar{\Psi}\gamma^{\mu}\partial_{\mu}\psi.$$

where $\psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$.

you get

$$i\,\psi_L^{\dagger}\,\bar{\sigma}^{\mu}\partial_{\mu}\psi_L$$

$$i\,\psi_R^{\dagger}\,\sigma^{\mu}\partial_{\mu}\psi_R$$

By Euler Lagrange you have

$$i\bar{\sigma}^{\mu}\cdot\partial_{\mu}\psi_L = 0 \qquad i\sigma^{\mu}\partial_{\mu}\psi_R = 0.$$

e) $U_+(p) = \begin{pmatrix} \sqrt{E-p} \\ \sqrt{E+p} \end{pmatrix} \chi_+(p) = \begin{pmatrix} 0 \\ \sqrt{2E} \end{pmatrix} \chi_+ p$ for $m=0$, $|E| = |p|$

$U_-(p) = \begin{pmatrix} \sqrt{2E} \\ 0 \end{pmatrix} \chi_-(p).$

$V_+(p) = \begin{pmatrix} 0 \\ \sqrt{2E} \end{pmatrix} \chi_+(p)$

$V_-(p) = \begin{pmatrix} \sqrt{2E} \\ 0 \end{pmatrix} \chi_-(p).$

We see only the the $U_-(p)$ and $V_-(p)$ ~~only satisfy the~~ are the only nonzero solutions to $i\bar{\sigma}^\mu \partial_\mu \Psi_L = 0.$

f). Now the particle has $-$ helicity (corresponds to $a^\dagger_-(p)|0\rangle$ which from 2a) has $-$ helicity). And the antiparticle has $+$ helicity (since $b^\dagger_-(p)|0\rangle$ has $+$ helicity). Having only 1 helicity state doesn't contradict results of part c, since for massless particles, you can never boost up so that you catch up to them because they travel at the speed of light. Therefore if the particle has $-$ helicity in one frame, it has $-$ helicity in all frames.