Homework #2 Solution Set

1. $\mathcal{L} = \psi^\dagger\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\right)\psi - \frac{\lambda}{2}:(\psi^\dagger\psi)^2:$

a) In HW#1, L is invariant under $\psi \to \psi e^{i\alpha}$ and $\psi^\dagger \to \psi^\dagger e^{-i\alpha}$. Thus by Noether's theorem there is a current, j_μ associated with symmetry such that $Q = \int d^3x\, j^0$ is conserved. In this problem one wishes to verify that $\frac{d}{dt}Q = 0$ using the equations of motion.

For the above Symmetry, using

$$\Delta\psi = i\psi, \quad \Delta\psi^\dagger = -i\psi^\dagger$$

$$j^0 = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)}\Delta\psi + :\frac{\partial\mathcal{L}}{\partial(\partial_0\psi^\dagger)}\Delta\psi^\dagger: = -2\hbar\,\psi^\dagger\psi \propto \psi^\dagger\psi$$

thus we have

$$Q = \int d^3x\, j^0 \propto \int d^3x\, \psi^\dagger\psi \quad \text{Should be Conserved.}$$

so

$$\frac{d}{dt}Q = \int d^3x\left[\left(\frac{d\psi^\dagger}{dt}\right)\psi + \psi^\dagger\frac{d}{dt}\psi\right]$$

Now $\mathcal{L}(\psi,\psi^\dagger,\partial_\mu\psi,\partial_\mu\psi^\dagger) = \psi^\dagger\, i\hbar\frac{\partial\psi}{\partial t} - \nabla\psi^\dagger\cdot\nabla\psi\,\frac{\hbar^2}{2m} - \frac{\lambda}{2}:(\psi^\dagger\psi)^2:$

So Euler Lagrange equations give

$$i\hbar\frac{\partial\psi^\dagger}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi^\dagger - \lambda\psi^\dagger\psi^\dagger\psi$$

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar^2}{2m}\nabla^2\psi + \lambda\psi^\dagger\psi\psi$$

so one has

$$\frac{dQ}{dt} \propto \int d^3x\left(-\frac{\hbar^2}{2m}(\nabla^2\psi)\psi - \cancel{\lambda\psi^\dagger\psi^\dagger\psi\psi} + \psi^\dagger\frac{\hbar^2}{2m}\nabla^2\psi + \cancel{\lambda\psi^\dagger\psi^\dagger\psi\psi}\right)$$

$$= \int d^3x\left(\frac{\hbar^2}{2m}\left(\psi^\dagger\nabla^2\psi - (\nabla^2\psi^\dagger)\psi\right)\right)$$

integrate by parts

$$\overset{\downarrow}{=}\ \underset{\substack{\| \\ 0}}{\cancel{\text{surface terms}}}\ -\frac{\hbar^2}{2m}\int d^3x\left(\nabla\psi^\dagger\cdot\nabla\psi - \nabla\psi^\dagger\cdot\nabla\psi\right) = 0.$$

Thus $\frac{dQ}{dt} = 0$ and Q is conserved.

(ii) $\psi(x) = \int d^3p\, e^{ip\cdot x} a(p).$ (neglecting constants)

$\psi^\dagger(x) = \int d^3p\, e^{-ip\cdot x} a^\dagger(p).$

So $Q = \int d^3x\, \psi^\dagger(x)\psi(x) = \int d^3x \int d^3p \int d^3p'\, e^{i(p-p')\cdot x} a^\dagger(p')a(p).$

$= \int d^3p \int d^3p' \; \delta^3(p-p') \quad a^\dagger(p')\, a(p)$

$= \int d^3p \; a^\dagger(p)\, a(p).$

but $a^\dagger(p)\, a(p)$ counts how many particles have momentum p, since $a^\dagger(p) a(p) \,|n_{p'}\rangle = n |n_{p'}\rangle \,\delta_{pp'}$

Thus $\int d^3p \, a^\dagger(p)\, a(p)$ counts all particles of system.

Thus since it's conserved This means # of particles conserved.

b) $\psi(x) \to \psi(x+a) \approx \psi(x) + \vec{a} \cdot \nabla \psi$

so $\Delta\psi = \nabla\psi \qquad \Delta\psi^\dagger = \nabla\psi^\dagger$

so $j^0 = i\hbar\,(\psi^\dagger \nabla\psi - (\nabla\psi^\dagger)\psi)$

so conserved quantity is

$Q = \int d^3x \; j^0 = i\hbar \int d^3x \; (\psi^\dagger \nabla\psi - (\nabla\psi^\dagger)\psi)$

$= 2i\hbar \int d^3 \psi^\dagger \nabla\psi \quad + \underbrace{\cancel{\text{surface terms}}}_{0} \;\propto \int d^3 \psi^\dagger \nabla\psi$

So

$\frac{dQ}{dt} = \int d^3x \left(\frac{\partial}{\partial t}\psi^\dagger\right) \nabla\psi + \psi^\dagger \nabla \frac{\partial}{\partial t}\psi$

$$\propto \int d^3x \left(-\frac{\hbar^2}{2m}(\nabla^2\psi^\dagger)\nabla\psi - \lambda\psi^\dagger\psi^\dagger\psi\nabla\psi + \psi^\dagger\frac{\hbar^2}{2m}\nabla(\nabla^2\psi) + \lambda\psi^\dagger\nabla(\psi^\dagger\psi\psi)\right)$$

$$= \int d^3x \left(-\frac{\hbar^2}{2m}\overbrace{\left((\nabla^2\psi^\dagger)\nabla\psi - \psi^\dagger\nabla(\nabla^2\psi)\right)}^{1} + \lambda\overbrace{\left(\psi^\dagger(\nabla\psi^\dagger)\psi\psi + 2\psi^\dagger\psi^\dagger\psi\nabla\psi\right.}^{2}\left. - \psi^\dagger\psi^\dagger\psi\nabla\psi\right)\right)$$

Now looking at term 1.

$$\int d^3x\left[(\nabla^2\psi^\dagger)\nabla\psi - \psi^\dagger\nabla(\nabla^2\psi)\right]$$

$$= \underset{0}{\underset{\parallel}{\text{Surface term}}} + \int d^3x\left(-\nabla\psi^\dagger\nabla^2\psi + \nabla\psi^\dagger\nabla^2\psi\right) = 0.$$

Looking at term 2 gives

$$\int d^3x\,\lambda\left(\psi^\dagger\psi^\dagger\psi\nabla\psi + \psi^\dagger(\nabla\psi^\dagger)\psi\psi\right)$$

Gauss's Law

$$= \frac{1}{2}\int d^3x\,\lambda\left(\nabla(\psi^\dagger\psi^\dagger\psi\psi)\right) = \frac{1}{2}\int_{\text{Surface}}\psi^\dagger\psi^\dagger\psi\psi\; n\cdot dS = 0.$$

Since ψ's are zero on surface.

thus

$$\frac{dQ}{dt} = 0.$$ and so Q is conserved.

(ii) $\nabla \psi(x) = \int d^3p \; a(p) \; \nabla e^{ip\cdot x} = \int d^3p \; a_p \; \vec{p} \; e^{ip\cdot x}$.

So $\int d^3x \; \psi^\dagger \nabla \psi = \int d^3x \int d^3p \int d^3p' \; a^\dagger(p) a(p') p' \; e^{i(p'-p)\cdot x}$.

$= \int d^3p \int d^3p' \; p' \; a^\dagger(p) a(p') \; \delta^3(p-p')$.

$= \int d^3p \; p \; \underbrace{a^\dagger(p) a(p)}_{\#\text{ operator}} \; = \text{total liner momentum}$.

Thus this corresponds to conservation of total liner momentum.

2. $\{ \pi(x), \phi(y) \} = \pi(x)\phi(y) + \phi(y)\pi(x) = i\,\delta^3(x-y)$.

So daggering both sides gives

$$\phi^\dagger(y)\, \pi^\dagger(x) + \pi^\dagger(x)\, \phi^\dagger(y) = -i\,\delta^3(x-y).$$

So $\boxed{\{ \pi^\dagger(x), \phi^\dagger(y) \} = -i\,\delta^3(x-y)}$

b) $\phi(x) = \int \frac{d^3p}{(2\pi)^3 \, 2E_p} \left(a(p) e^{ip\cdot x} + b^\dagger(p) e^{-ip\cdot x} \right)$

$\pi(x) = \int \frac{d^3p}{(2\pi)^3} \left(-\frac{i}{2}\right) \left(b(p) e^{ip\cdot x} - a^\dagger(p) e^{-ip\cdot x} \right)$

This gives.

$$a(p) = \int d^3x\, e^{-ip\cdot x}\left(\phi(x)E_p + i\pi^\dagger(x)\right)$$

$$a^\dagger(p) = \int d^3x\, e^{ip\cdot x}\left(\phi^\dagger(x)E_p - i\pi(x)\right)$$

$$b(p) = \int d^3x\, e^{-ip\cdot x}\left(\phi^\dagger(x)E_p + i\pi(x)\right)$$

$$b^\dagger(p) = \int d^3x\, e^{ip\cdot x}\left(\phi(x)E_p - i\pi^\dagger(x)\right)$$

So

$$\{a(p), a^\dagger(p')\} = \int d^3x \int d^3y\, e^{ip'\cdot y} e^{-ip\cdot x}\Big(\overbrace{\{\phi^\dagger(y), \phi(x)\}E_p^2 + \{\pi^\dagger(x), \pi(y)\}}^{\text{takes zero}}$$

$$+ \{\phi^\dagger(y)\, \pi(x)\}\, iE_p - i\{\pi(y), \phi(x)\}E_p\Big)$$

$$= \int d^3x \int d^3y\, e^{ip'\cdot y} e^{-ip\cdot x}\, 2E_p\, \delta(x-y)$$

$$= 2E_p \int d^3y\, e^{i(p-p')\cdot y} = 2E_p (2\pi)^3 \delta(p-p')$$

$$\{b(p)\, b^\dagger(p')\} = \int d^3x \int d^3y\, e^{-ip\cdot x} e^{ip'\cdot y}\Big(\overbrace{\{\phi^\dagger(x), \phi(x)\}}^{0}E_p^2 + \overbrace{\{\pi^\dagger(x), \pi(y)\}}^{0}$$

$$+ \{\phi^\dagger(y), \pi^\dagger(x)\}(-i)E_p + i\{\pi(y), \phi(x)\}.$$

$$= -2E_p \int d^3x \int d^3y\, e^{ip'\cdot y} e^{-ip\cdot x}\, \delta(x-y)$$

$$= \boxed{-2E_p (2\pi)^3\, \delta(p-p')}$$

and.

$$\{a(p)\, b(p')\} = \int d^3x \int dy^3\, e^{-ip\cdot x} e^{ip'\cdot y} \Big(E_p^2 \underbrace{\{\phi(x)\, \phi^\dagger(y)\}}_{=0} - \underbrace{\{\Pi^\dagger(x), \Pi y\}}_{=0}$$

$$+ i \underbrace{\{\phi(x), \Pi(y)\}}_{-\delta^3(x-y)E_p} E_p + i \underbrace{\{\Pi^\dagger(x)\, \phi^\dagger(y)\}}_{+\delta^3(x-y)E_p} E_p \qquad = 0.$$

and

$$\{a(p), b^\dagger(p')\} = \int d^3x \int d^3y\, e^{-ip\cdot x} e^{ip'\cdot y} \Big(iE_p \big(\{\phi(x)\, \Pi^\dagger(y)\} - \{\phi(y)\Pi^\dagger(x)\}\big)\Big) = 0$$

since x, y just dummy variables.

Now consider state

$b^\dagger(p)|0\rangle$. Norm is given by since $b(p)|0\rangle = 0$

$$\langle 0|\, b(p)\, b^\dagger(p)\, |0\rangle = \langle 0|\, b(p)\, b^\dagger(p) + b^\dagger(p)\, b(p)\, |0\rangle$$

$$= \langle 0|\, \{b(p), b^\dagger(p)\}\, |0\rangle = -2E_p\, \delta^3(0) < 0.$$

c) Take $\{b(p)\, b^\dagger(p')\} = (+)2E_p (2\pi)^3 \delta^3(p-p')$

So look at.

$$\{\phi(x), \phi^\dagger(y)\} = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \frac{e^{ip\cdot x} e^{-ip'\cdot y}}{4E_pE_{p'}} \{a(p), a^\dagger(p')\}$$

$$+ \underline{\{ b(p') b^\dagger(p) \} e^{ip'\cdot y}} e^{-ip\cdot x} \Big)$$

$$= \int \frac{d^3p}{(2\pi)^3} \int d^3p' \frac{\cancel{4} E_p \delta^3(p-p')}{\cancel{4} E_p E_{p'}} \left(e^{ip\cdot x} e^{-ip'\cdot y} + e^{ip'\cdot y} e^{-ip\cdot x} \right).$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{e^{ip\cdot(x-y)} + e^{-ip\cdot(y-x)}}{E_p}$$

$$= D(x-y) + D(y-x)$$

For spacelike separation, one no longer gets a cancellation since you add $D(x-y)$ and $D(y-x)$ instead of subtract. Thus $\{\phi^\dagger(x), \phi(y)\} \neq 0$ for spacelike separation, so causality is violated.

d) If $\{\phi(x)\, \phi(y)\} = 0$.

then $\phi(x)\phi(x) + \phi(x)\phi(x) = 0$

so $\phi^2(x) = 0$.

Let ϵ^μ be a vector in the direction μ.

$$\partial_\mu \phi(x) \equiv \lim_{|\epsilon| \to 0} \frac{\phi(x+\epsilon^\mu) - \phi(x)}{\epsilon^\mu}$$

So

$$\partial_\mu \phi(x) \partial^\mu \phi(x) = \lim_{|\epsilon_\mu| \to 0} \left(\frac{\phi(x+\epsilon^\mu) - \phi(x)}{\epsilon^\mu} \right) \left(\frac{\phi(x+\epsilon^\mu) - \phi(x)}{\epsilon_\mu} \right)$$

$$= \lim_{|\epsilon_\mu| \to 0} \frac{1}{\epsilon^\mu \epsilon_\mu} \Big(\cancelto{0}{\phi(x+\epsilon^\mu)\phi(x+\epsilon^\mu)} + \cancelto{0}{\phi(x)\phi(x)} - \cancelto{0}{\{\phi(x+\epsilon^\mu), \phi(x)\}} \Big) = 0$$

$\therefore \quad \boxed{S = 0}$