HW #2, due Sep 17

1. There are conserved quantitites associated with the transformations in HW #1, **1.** (i) Show that they are conserved using the equation of motion, and (ii) discuss the physical meaning of the conserved quantities by rewriting them in terms of the mode operators $a(\vec{p})$.

(1) $\int d^3x\, \psi^\dagger \psi$

(2) $\int d^3x\, \psi^\dagger(-i\hbar\vec{\nabla})\psi$

Rem The conserved quantity associated with the Galilean boost is $\int d^3x \psi^\dagger(m\vec{x} + it\hbar\vec{\nabla})\psi$, and the conservation of this quantity describes the motion of the center of mass.

2. The spin statistics relation. Consider the action

$$S = \int d^4x\, (\partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi). \qquad (1)$$

for a complex Klein–Gordon field ϕ. Let us see whether we can consistently quantize the theory using Fermi statistics. The conjugate momenta to $\phi(x)$ and $\phi^\dagger(x)$ are the same as obtained in the class.

(1) Impose an anti-commutation relation $\{\pi(x), \phi(y)\} = i\delta^3(\vec{x} - \vec{y})$. Determine the anti-commutator $\{\pi^\dagger(x), \phi^\dagger(y)\}$.

(2) By using the same mode expansion as in the class, obtain anti-commutation relations among the $a(p)$, $b(p)$ and their hermitean conjugates. Show that $b^\dagger(p)$ creates a state with a negative norm, *i.e.*, $|b^\dagger(p)|0\rangle|^2 < 0$.

(3) Change the anti-commutation relation for $b(p)$ and $b^\dagger(p)$ to the usual one *by hand* so that you have a Hilbert space with positive definite norms. Derive the anti-commutator $\{\phi(x), \phi^\dagger(y)\}$ using the modified anti-commutation relations among the creation and annihilation operators. Show that it violates causality.

(4) Show that a similar action for a *real* scalar field vanishes identically if $\phi(x)$ is anti-commuting, *i.e.*, $\{\phi(x), \phi(y)\} = 0$.