HW #11, due Nov 19

1. Charge Radius and Lamb Shift. The vertex correction $F_1(q^2)$ is effectively a "momentum-transfer dependent charge." In other words, it gives a charge density profile once Fourier-transformed back to the coordinate space.

(a) Expand $F_1(q^2) = 1 + q^2 F_1'(0) + O(q^2)^2$. Obtain $F_1'(0)$.

(b) The "charge density profile" is given by

$$\rho(\vec{r}) = \int \frac{d^3\vec{q}}{(2\pi)^3} F_1(-\vec{q}^2) e^{i\vec{q}\cdot\vec{r}}. \tag{1}$$

Here and below $q^0 = 0$ and $q^2 = -\vec{q}^2$. Show that the effective "charge radius" is

$$\langle(\delta\vec{r})^2\rangle \equiv \int d^3\vec{r}\, \vec{r}^2 \rho(\vec{r}) = -\left(\frac{\partial}{\partial\vec{q}}\right)^2 F_1(-\vec{q}^2). \tag{2}$$

Obtain the charge radius.

(c) In the presence of the Coulomb potential, $V = -\frac{Z\alpha}{r}$, show that the fluctuation in $\vec{r} = \vec{r}_0 + \delta\vec{r}$ leads to a change in the potential given by

$$\delta V = \frac{1}{6}\langle(\delta\vec{r})^2\rangle 4\pi Z\alpha\delta^3(\vec{r}_0). \tag{3}$$

Note that $\langle\delta\vec{r}\rangle = 0$, $\langle\delta r^i \delta r^j\rangle = \frac{1}{3}\delta^{ij}\langle(\delta\vec{r})^2\rangle$.

(d) Calculate the shift in the $2S^{1/2}$ energy level in the hydrogen atom due to the above additional term in the potential. (Non-relativistic wave function is enough for our purpose.)

(e) Dependence on the infrared cutoff μ is an artifact of having used plane waves in calculating the vertex corrections without considering bound states. On intuitive grounds, the finite size of the bound state wave function and/or the binding energy lead to a costant "off-shellness" of the virtual electrons and should provide a natural infrared cutoff. Therefore, we expect $(Z\alpha)^2 m \lesssim \mu \lesssim (Z\alpha)m$. Using this range, estimate the size of the shift in the $2S^{1/2}$ energy level in terms of the frequency. Compare it to the observed Lamb shift.

Rem A complete treatement of Lamb shift is quite non-trivial, which involves the bound state wave functions in the loop diagram, re-evaluation of the self-energy diagram with bound-state effects, the vacuum polarization effect, and $g-2$. But the effect discussed qualitatively in this homework is the dominant one.