Take-home Final Exam, due Dec 10, 12:40 pm sharp

1. Pair production of electrons from two photons. There are photons in cosmic rays of extragalactic origin with incredibly high energies, well beyond TeV. The extremely high energy ones, however, do not reach the Earth because they scatter with the photons in the microwave background to produce electron-positron pairs. Let us study such an effect. We denote the four-momenta of two initial photons by q_1 and q_2, and the final electron (positron) by p ($\bar{p}$).

(a) Kinematics question. What is the minimum energy of the incident photon in order to produce an electron-positron pair? Assume a typical photon in the cosmic microwave background has an energy of 3×10^{-4} eV. (Actually, the minimum energy is much lower than this estimate because of the tail in the Planck distribution.)

(b) Draw *two* Feynman diagrams for $\gamma\gamma \to e^+e^-$.

(c) Make an estimate of the cross section at (i) threshold $s \sim 4m^2$, and at (ii) high energy $s \gg 4m^2$, using the simple method I described in the class. You are supposed to obtain factors of coupling constant, π and mass (i) or energy (ii) dependence correctly.

(d) Write down the amplitudes for each Feynman diagrams. What is the relative sign between the two amplitudes?

(e) Obtain the spin-summed squared amplitude. (Actually, you don't need to calculate this; use a trick to obtain it from one of the cross sections we calculated in the class.)

(f) Calculate the total cross section as a function of s. (The result is quite messy.) Plot it as a function of s.

(g) Take a limit $s \to 4m^2$ and compare the expression with what you obtained in (c).

(h) Assume that the cosmic microwave background consists of photons with energy $E = kT_0$ (which is, of course, an oversimplification) and number density $n = 2(\zeta(3)/\pi^2)T_0^3$ with $T_0 = 2.726$ K. What is the mean free path of the incident photon in the units of kpc? An order of magnitude estimate is enough.

2. beta-function in ϕ^4 theory. The Lagrangian of the ϕ^4 theory is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4. \quad (1)$$

Here, ϕ is a real Klein–Gordon scalar field of mass m. The only interaction in this theory is the vertex among four scalar lines with the Feynman rule $-i\lambda$.

(a) Write down the one-loop amplitude for the scalar two-point function. (Hint: you need to be careful about combinatoric factors. It is the easiest if you go back to the Wick's theorm. You will find a multiplicative factor of 1/2 compared to the normalization you expect naively from the Feynman rule.) Show that this amplitude does not generate wave-function renormalization. This diagram gives quadratically (!) divergent mass renormalization.

(b) Write down the one-loop amplitude for the scalar four-point function. (Hint: there are three diagrams.)

(c) We are interested only in the ultraviolet divergent part to calculate the four-point function. Therefore, you can set the four-momenta of external lines vanishing. To obtain the logarithmically divergent piece, you can do either of the following two: (1) subtract the same amplitude with Pauli–Villars mass M, or (2) do the Wick rotation first, and cutoff the momentum integration at $k_E^2 < M^2$. In either case, you should keep the mass m in your calculation.

(d) Calculate the beta-function $\beta(\lambda) = -M^2 d\lambda/dM^2$ where you differentiate λ with the bare coupling λ_0 fixed.

(e) Use the dimensional regularization to perform the loop integral, with all external four-momenta neglected but with the mass of the scalar kept. Write the four-point amplitude using the bare couping λ_0 and an arbitrary dimensionful parameter μ. (The expression is $\lambda = \lambda_0 \mu^{-2\epsilon} + O(\lambda_0 \mu^{-2\epsilon})^2$ in $D = 4 - 2\epsilon$ dimension. You have calculated the second piece.) Calculate $\beta = \mu^2 d\lambda/d\mu^2$ where you differentiate λ with the bare coupling λ_0 fixed, and the limit $\epsilon \to 0$ is taken after the differentiation. Terms of $O(\lambda^3)$ are neglected in β.

(f) Integrate the beta function to obtain the behavior of the running coupling constant $\lambda(Q^2)$. Is it asymptotically-free (decreasing with energy) or infrared-free (increasing with energy)?

3. Electron Electric Dipole Moment In supersymmetric theories, it is possible that the electron acquires an electric dipole moment from the loop diagram of selectrons (superpartner of electron) and photino (superpartner of photon). The kinetic terms are as usual,

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - \mu)\psi + \bar{\lambda}(i\not{\partial} - m)\lambda + D_\mu\phi_L^* D^\mu\phi_L - M_L^2\phi_L^*\phi_L + D_\mu\phi_R^* D^\mu\phi_R - M_R^2\phi_R^*\phi_R \tag{2}$$

where ψ is the electron field, λ the photino field (spinor), and $\phi_{L,R}$ the left-handed (right-handed) selectron field (Klein–Gordon fields). The QED Feynman rules for the Klein–Gordon fields are given in Problem 9.1 of the book. The new interactions of the photino, selectron and electron are given by

$$\mathcal{L}_{int} = -Am\,\phi_L^*\phi_R - A^*m\,\phi_R^*\phi_L$$
$$-\sqrt{2}e\,\bar{\lambda}\left(\frac{1+\gamma_5}{2}\phi_R^* + \frac{1-\gamma_5}{2}\phi_L^*\right)\psi - \sqrt{2}e\,\bar{\psi}\left(\frac{1+\gamma_5}{2}\phi_L + \frac{1-\gamma_5}{2}\phi_R\right)\lambda. \tag{3}$$

Calculate the one-loop amplitude of the type

$$u(\bar{p}')q_\nu\sigma^{\mu\nu}\gamma_5 u(p), \tag{4}$$

assuming $M_L = M_R = \mu = A = M_{SUSY} \gg m$. Show that this amplitude corresponds to the electric dipole moment of the electron. Using the result by Gene Commins' group $d_e = (0.18 \pm 0.12 \pm 0.10) \times 10^{-26} e$ cm, place a lower bound on the superparticle mass scale M_{SUSY}. It simplifies the calculation drastically if you treat the terms with A parameter as "interactions" with the Feynman rule $-iAm$ and $-iA^*m$ rather than the mass terms, and use them only once in the diagram.