Perturbation Theory

To make up my embarrassment, I outline the derivation of the time-dependent perturbation theory below. I thank Craig and James for help.

Step 0. We start with operators $O(t_0)$ and states $|\psi(t_0)\rangle$ defined at a reference time t_0. In Schrödinger picture, states evolve in time $|\psi(t)\rangle_S = U_F(t,t_0)|\psi(t_0)\rangle$ where $U_F(t,t_0) = e^{-iH(t-t_0)}$ is the time-evolution operator with the full Hamiltonian. The operators do not depend on time: $O_S(t) = O(t_0)$. On the other hand in Heisenberg picture, states do not evolve in time $|\psi(t)\rangle_H = |\psi(t_0)\rangle$, but the operators do: $O_H(t) = U_F(t,t_0)^\dagger \mathcal{O}(t_0) U_F(t,t_0)$. It is easy to check that matrix elements are the same in either picture: $\langle\psi_2(t)|_H \mathcal{O}_H(t)|\psi_1(t)\rangle_H = \langle\psi_2(t)|_S \mathcal{O}_S(t)|\psi_1(t)\rangle_S$. If we suppress the subscript in states or operators below, they are in Heisenberg picture.

Step 1. The full Hamiltonian is divided into the unperturbed piece H_0 and the interaction H_{int} as $H = H_0 + H_{int}$. We define the operators in the interaction picture by

$$O_I(t) \equiv U_0(t,t_0)^\dagger O(t_0) U_0(t,t_0), \tag{1}$$

where $U_0(t,t_0) = e^{-iH_0(t-t_0)}$ is the time evolution operator in the unperturbed theory. To keep the matrix elements the same as those in other pictures, time evolution of the states is fixed to be

$$|\psi(t)\rangle_I = U_0(t,t_0)^\dagger U_F(t,t_0)|\psi(t_0)\rangle \equiv U_I(t,t_0)|\psi(t_0)\rangle. \tag{2}$$

Note that t can be either later or earlier than t_0.

Step 2. We rewrite $U_I(t,t_0) = U_0(t,t_0)^\dagger U_F(t,t_0)$ using time-ordered products. We solve the differential equation,

$$i\frac{\partial}{\partial t}U_I(t,t_0) = U_0(t,t_0)^\dagger H_{int} U_0(t,t_0) U_I(t,t_0) \equiv H_I(t) U_I(t,t_0), \tag{3}$$

and find

$$U_I(t,t_0) = Te^{-i\int_{t_0}^{t} H_I(t')dt'}. \tag{4}$$

Here again t can be either later or earlier than t_0.

Step 3.[*] We generalize the definition of $U_I(t,t_0)$ to arbitrary arguments $U_I(t_2,t_1)$ by *demanding* the following nice property

$$U_I(t_3,t_2)U_I(t_2,t_1) = U_I(t_3,t_1). \tag{5}$$

[*]This is the step I've missed in the class.

By choosing a particular case of $t_3 = t_1 = t_0$, and using the fact $U_I(t_0, t_0) = 1$, we find

$$U_I(t_0, t) = U_I(t, t_0)^\dagger = U_F(t, t_0)^\dagger U_0(t, t_0) = U_F(t_0, t) U_0(t_0, t)^\dagger. \tag{6}$$

Using again Eq. (5) for $t_2 = t_0$ this time, we find

$$\begin{aligned} U_I(t_3, t_1) &= U_I(t_3, t_0) U_I(t_0, t_1) \\ &= \left[U_0(t_3, t_0)^\dagger U_F(t_3, t_0)\right] \left[U_F(t_0, t_1) U_0(t_0, t_1)^\dagger\right] = U_0(t_0, t_3) U_F(t_3, t_1) U_0(t_1, t_0). \end{aligned} \tag{7}$$

This defines U_I for arbitrary arguments, and it is easy to check that this expression satisfies the property we demanded Eq. (5).

Step 4. We show that $U_I(t_2, t_1)$ defined above (7) can be written as[†]

$$U_I(t_2, t_1) = T e^{-i \int_{t_1}^{t_2} H_I(t') dt'}. \tag{8}$$

First, $U_I(t, t_1) = U_I(t, t_0) U_I(t_0, t_1)$ follows the same differential equation Eq. (3) under t derivative because the $U_I(t_0, t_1)$ piece does not depend on t. Second, r.h.s. of Eq. (8) also follows the same differential equation, which can be checked explicitly by using the Taylor expansion of the exponential. Third, both sides of the equation reduce to 1 (the identity operator) when $t_2 \to t_1$. Therefore, l.h.s. and r.h.s. of the equation satisfy the same first-order differential equation and has the same boundary condition, and are hence the same.

Step 5. We rewrite the ground state of the full Hamiltonian $H|\Omega\rangle = E_\Omega |\Omega\rangle$ in terms of the ground state of the unperturbed Hamiltonian $H_0|0\rangle = E_0|0\rangle$. First,

$$\begin{aligned} U_F(t_0, -T)|0(t_0)\rangle &= U_F(t_0, -T) \left[|\Omega(t_0)\rangle\langle\Omega|0\rangle + \sum_i |i(t_0)\rangle\langle i|0\rangle\right] \\ &= e^{-iE_\Omega(t_0+T)}|\Omega(t_0)\rangle\langle\Omega|0\rangle + \sum_i e^{-iE_i(t_0+T)}|i(t_0)\rangle\langle i|0\rangle. \end{aligned} \tag{9}$$

Now we take the limit $T \to \infty(1 - i\epsilon)$ where all excited states $|i\rangle$ get suppressed exponentially relative to the ground state:

$$U_F(t_0, -T)|0(t_0)\rangle = e^{-iE_\Omega(t_0+T)}|\Omega(t_0)\rangle\langle\Omega|0\rangle. \tag{10}$$

Here and hereafter, all equalities hold only in the limit. The l.h.s. of the equation can be rewritten as

$$U_F(t_0, -T)|0(t_0)\rangle = U_I(t_0, -T) U_0(-T, t_0)^\dagger |0(t_0)\rangle = U_I(t_0, -T)|0(t_0)\rangle e^{-iE_0(t_0+T)}, \tag{11}$$

[†]We could have adopted this as the definition of U_I for arbitrary arguments, and shown Eq. (5). But I find this logic more transparent.

and putting them together,

$$|\Omega(t_0)\rangle = \frac{e^{-i(E_0-E_\Omega)(t_0+T)}}{\langle\Omega|0\rangle} U_I(t_0,-T)|0(t_0)\rangle. \tag{12}$$

Step 6. Following exactly the same steps, we find

$$\langle 0(t_0)|U_F(T,t_0) = \langle 0|\Omega\rangle\langle\Omega(t_0)|e^{-iE_\Omega(T-t_0)} \tag{13}$$

in the limit $T \to \infty(1-i\epsilon)$. The l.h.s. of the equation can be rewritten as

$$\langle 0(t_0)|U_F(T,t_0) = \langle 0(t_0)|U_0(T,t_0)U_I(T,t_0) = \langle 0(t_0)|U_I(T,t_0)e^{-iE_0(T-t_0)}, \tag{14}$$

and putting them together,

$$\langle\Omega(t_0)| = \frac{e^{-i(E_0-E_\Omega)(T-t_0)}}{\langle 0|\Omega\rangle} \langle 0(t_0)|U_I(T,t_0). \tag{15}$$

Step 7. Combining Eqs. (12) and (15), we find

$$\begin{aligned} 1 = \langle\Omega|\Omega\rangle &= \frac{e^{-i(E_0-E_\Omega)(T-t_0)}e^{-i(E_0-E_\Omega)(t_0+T)}}{|\langle 0|\Omega\rangle|^2} \langle 0(t_0)|U_I(T,t_0)U_I(t_0,-T)|0(t_0)\rangle \\ &= \frac{e^{-i(E_0-E_\Omega)2T}}{|\langle 0|\Omega\rangle|^2} \langle 0(t_0)|U_I(T,-T)|0(t_0)\rangle. \end{aligned} \tag{16}$$

Step 8. We now look at time-ordered products of operators in Heisenberg picture:

$$\mathcal{O} = TO_1(t_1)O_2(t_2)\cdots O_n(t_n) \tag{17}$$

Of course the operators also in general depend on positions in space ($\vec{x}_1$ etc), but I suppressed them to simplify the expression. The operators in the interaction picture Eq. (1) are related to those in Heisenberg picture as

$$\begin{aligned} O(t) &= U_F(t,t_0)^\dagger O(t_0) U_F(t,t_0) \\ &= U_F(t,t_0)^\dagger \left[U_0(t,t_0)O_I(t)U_0(t,t_0)^\dagger\right] U_F(t,t_0) \\ &= U_I(t,t_0)^\dagger O_I(t_0) U_I(t,t_0). \end{aligned} \tag{18}$$

We rewrite O in Eq. (17) in the interaction picture. We consider the case $t_1 > t_2 > \cdots > t_n$. Then,

$$\begin{aligned} \mathcal{O} &= O_1(t_1)O_2(t_2)\cdots O_n(t_n) \\ &= U_I(t_1,t_0)^\dagger O_{1I}(t_1)U_I(t_1,t_0)U_I(t_2,t_0)^\dagger O_{2I}(t_2)U_I(t_2,t_0) \\ &\quad \cdots U_I(t_n,t_0)^\dagger O_{nI}(t_n)U_I(t_n,t_0) \\ &= U_I(t_0,t_1)O_{1I}(t_1)U_I(t_1,t_2)O_{2I}(t_2)U_I(t_2,t_3)\cdots U_I(t_{n-1},t_n)O_{nI}(t_n)U_I(t_n,t_0), \end{aligned} \tag{19}$$

where Eqs. (5), (6) were used in the last step.

Step 9. Now we are in the position to rewrite the correlation functions using the operators in the interaction picture. We would like to compute the correlation function (in Heisenberg picture)

$$G = \langle \Omega | T O_1(t_1) O_2(t_2) \cdots O_n(t_n) | \Omega \rangle = \langle \Omega | \mathcal{O} | \Omega \rangle . \tag{20}$$

Using Eqs. (12), (15), and (19), we find

$$\begin{aligned} G &= \frac{e^{-i(E_0 - E_\Omega)(T - t_0)}}{\langle 0 | \Omega \rangle} \langle 0(t_0) | U_I(T, t_0) U_I(t_0, t_1) O_{1I}(t_1) U_I(t_1, t_2) O_{2I}(t_2) U_I(t_2, t_3) \\ &\quad \cdots U_I(t_{n-1}, t_n) O_{nI}(t_n) U_I(t_n, t_0) U_I(t_0, -T) | 0(t_0) \rangle \frac{e^{-i(E_0 - E_\Omega)(t_0 + T)}}{\langle \Omega | 0 \rangle} \\ &= \frac{e^{-i(E_0 - E_\Omega)2T}}{|\langle 0 | \Omega \rangle|^2} \langle 0(t_0) | U_I(T, t_1) O_{1I}(t_1) U_I(t_1, t_2) O_{2I}(t_2) U_I(t_2, t_3) \\ &\quad \cdots U_I(t_{n-1}, t_n) O_{nI}(t_n) U_I(t_n, -T) | 0(t_0) \rangle . \end{aligned} \tag{21}$$

Now recalling that we took the specific case of $t_1 > t_2 > \cdots > t_n$ and each $U_I(t_i, t_{i+1}) = T e^{-i \int_{t_{i+1}}^{t_i} H_I(t') dt'}$, we can write it in a compact form,

$$G = \frac{e^{-i(E_0 - E_\Omega)2T}}{|\langle 0 | \Omega \rangle|^2} \langle 0(t_0) | T O_{1I}(t_1) O_{2I}(t_2) \cdots O_{nI}(t_n) e^{-i \int_{-T}^{T} H_I(t') dt'} | 0(t_0) \rangle . \tag{22}$$

Finally using Eq. (16) we find Dyson's formula,

$$G = \frac{\langle 0(t_0) | T O_{1I}(t_1) O_{2I}(t_2) \cdots O_{nI}(t_n) e^{-i \int_{-T}^{T} H_I(t') dt'} | 0(t_0) \rangle}{\langle 0(t_0) | T e^{-i \int_{-T}^{T} H_I(t') dt'} | 0(t_0) \rangle} . \tag{23}$$

Even though we derived this formula for $t_1 > t_2 > \cdots > t_n$, this expression does not depend on this particular order because everything is time-ordered. This proves Dyson's formula.