HW #11

1. Bogoliubov Transformation

(a)

This is a straight-forward algebra
$[b, b^\dagger] = [a \cosh\eta + a^\dagger \sinh\eta,\ a^\dagger\, c\,\mathrm{osh}\,\eta + a \sinh\eta]$
$= [a, a^\dagger] \cosh^2\eta + [a^\dagger, a] \sinh^2\eta$
$= \cosh^2\eta - \sinh^2\eta$
$= 1$

(b)

Using $b, b^\dagger$ defined in (a), we can solve for $a, a^\dagger$ as
$a = b \cosh\eta - b^\dagger \sinh\eta$
$a^\dagger = b^\dagger \cosh\eta - b \sinh\eta$
Substituing them into the Hamiltonian ,we find
$H = \hbar\,\omega\, a^\dagger a + \frac{1}{2} V(a\,a + a^\dagger a^\dagger)$
$= \hbar\,\omega(b^\dagger \cosh\eta - b \sinh\eta)(b \cosh\eta - b^\dagger \sinh\eta)$
$+ \frac{1}{2} V\left((b^\dagger \cosh\eta - b \sinh\eta)^2 + (b \cosh\eta - b^\dagger \sinh\eta)^2\right)$
$= b^\dagger b\,(\hbar\,\omega \cosh^2\eta - V \cosh\eta \sinh\eta)$
$+ b\, b^\dagger (\hbar\,\omega \sinh^2\eta - V \cosh\eta \sinh\eta)$
$+ b\, b(-\hbar\,\omega \sinh\eta \cosh\eta + \frac{1}{2} V\,(\sinh^2\eta + \cosh^2\eta))$
$+ b^\dagger b^\dagger (-\hbar\,\omega \sinh\eta \cosh\eta + \frac{1}{2} V(\cosh^2\eta + \sinh^2\eta))$
Because η is a free parameter, we can choose it at our convenience. We choose η such that the coefficients of $b\,b$ and $b^\dagger b^\dagger$ vanish,
$-\hbar\,\omega \sinh\eta \cosh\eta + \frac{1}{2} V(\cosh^2\eta + \sinh^2\eta) = 0.$
$-\hbar\,\omega \sinh 2\,\eta + V \cosh 2\,\eta = 0.$
and hence
$\eta = \frac{1}{2} \operatorname{arctanh} \frac{V}{\hbar\,\omega}$, or $e^{4\,\eta} = \frac{\hbar\,\omega + V}{\hbar\,\omega - V}$.
With this choice, the Hamiltonian has the familiar form
$H = b^\dagger b\,(\hbar\,\omega \cosh^2\eta - V \cosh\eta \sinh\eta)$
$+ b\, b^\dagger (\hbar\,\omega \sinh^2\eta - V \cosh\eta \sinh\eta)$
$= b^\dagger b(\hbar\,\omega(\cosh^2\eta + \sinh^2\eta) - 2\,V \cosh\eta \sinh\eta) + (\hbar\,\omega \sinh^2\eta - V \cosh\eta \sinh\eta)$
$= b^\dagger b \sqrt{(\hbar\,\omega)^2 - V^2} + \frac{1}{2}\left(\sqrt{(\hbar\,\omega)^2 - V^2} - \hbar\,\omega\right)$
We find that the ground state is annihilated by b, its energy is $\frac{1}{2}\left(\sqrt{(\hbar\,\omega)^2 - V^2} - \hbar\,\omega\right)$, and the excitated states are obtained by acting $b^\dagger$ each time raising the energy by $\sqrt{(\hbar\,\omega)^2 - V^2}$.

The system does not have a stable ground state if $V > \hbar\,\omega$.

(c)

We use the Hausdorff formula,
$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots$
In our case, $A = (a\,a - a^\dagger a^\dagger)\,\eta/2$, $B = a$.

Let us first calculate the multiple commutators.
$B = a$
$[A, B] = \frac{\eta}{2}[a\,a - a^\dagger a^\dagger, a] = \eta\, a^\dagger$.
$[A, [A, B]] = \frac{\eta}{2}[a\,a - a^\dagger a^\dagger, \eta\, a^\dagger] = \eta^2 a$.
$[A, [A, [A, B]]] = \frac{\eta}{2}[a\,a - a^\dagger a^\dagger, \eta^2 a] = \eta^3 a^\dagger$
etc. By inspection, it is clear that the $2n$-th commutator is $\eta^{2n} a$, while the $2n+1$-th commutator $\eta^{2n+1} a^\dagger$.

Now back to the Hausdorff formula,
$e^A B e^{-A} = a + \eta\, a^\dagger + \frac{1}{2!}\eta^2 a + \frac{1}{3!}\eta^3 a^\dagger + \cdots$
$= a \cosh\eta + a^\dagger \sinh\eta = b$
Putting them together, we find $b = U a U^{-1}$ indeed.

We can also see that it is a unitarity operator. The exponent is anti-Hermitian,
$A^\dagger = \left(\frac{\eta}{2}(a\,a - a^\dagger a^\dagger)\right)^\dagger = \frac{\eta}{2}(a^\dagger a^\dagger - a\,a) = -\frac{\eta}{2}(a\,a - a^\dagger a^\dagger) = -A.$
Therefore, $U^\dagger = (e^A)^\dagger = e^{A^\dagger} = e^{-A} = U^{-1}$, and hence it is unitary.

(d)

The ground state is annihilated by b, and hence $U a U^{-1} \,|\,\mathrm{gs}\rangle = 0$.
Acting U^{-1} on both sides, we find $a(U^{-1}\,|\,\mathrm{gs}\rangle) = 0$, namely that the state $U^{-1}\,|\,\mathrm{gs}\rangle$ is annihilated by the operator a. Therefore,
$U^{-1}\,|\,\mathrm{gs}\rangle = |\,0\rangle$,
and hence
$|\,\mathrm{gs}\rangle = U\,|\,0\rangle$.
Because U is a power series in $\frac{\eta}{2}(a\,a - a^\dagger a^\dagger)$ which changes numbers by two units, we find that $|\,\mathrm{gs}\rangle$ can be expanded as
$|\,\mathrm{gs}\rangle = \sum_{n=0}^{\infty} c_{2n}\,|\,2n\rangle$

Using the fact that it is annihilated by b, $(a\cosh\eta + a^\dagger \sinh\eta)\,|\,\mathrm{gs}\rangle = 0$, we find
$b\,|\,\mathrm{gs}\rangle = (a\cosh\eta + a^\dagger \sinh\eta)\sum_{n=0}^{\infty} c_{2n}\,|\,2n\rangle$
$= \sum_{n=0}^{\infty} c_{2n}\left(\sqrt{2n}\,|\,2n-1\rangle \cosh\eta + \sqrt{2n+1}\,|\,2n+1\rangle \sinh\eta\right)$
$= \sum_{n=0}^{\infty}\left(c_{2n+2}\sqrt{2n+2}\,\cosh\eta + c_{2n}\sqrt{2n+1}\,\sinh\eta\right)|\,2n+1\rangle = 0.$
It means
$c_{2n+2} = c_{2n}\left(-\sqrt{\frac{2n+1}{2n+2}}\,\tanh\eta\right).$
Therefore,
$c_{2n} = c_0(-1)^n \sqrt{\frac{(2n-1)!!}{(2n)!!}}\,\tanh^n\eta$
The norm is
$1 = c_0{}^2 \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!}\tanh^{2n}\eta$
Rewriting it with the Gamma function,
$c_0{}^{-2} = \sum_{n=0}^{\infty} \frac{2^n\,\Gamma(n+1/2)/\sqrt{\pi}}{2^n\,\Gamma(n+1)}\tanh^{2n}\eta = \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)}{\Gamma(n+1)}\tanh^{2n}\eta$
Mathematica gives this infinite series

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In[28]:= Sum[Gamma[n + 1/2]/Gamma[n + 1] x^n, {n, 0, ∞}]
```

Out[28]= $\frac{\sqrt{\pi}}{\sqrt{1-x}}$

Therefore,
$c_0{}^{-2} = \frac{1}{\sqrt{\pi}} \frac{\sqrt{\pi}}{\sqrt{1-\tanh^2 \eta}} = (1-\tanh^2 \eta)^{-1/2}$
and hence
$c_0 = (1-\tanh^2 \eta)^{1/4}$.

Putting everything together, the ground state of the Hamiltonian is given in terms of the number states
$|\,\mathrm{gs}\rangle = (1-\tanh^2 \eta)^{1/4} \sum_{n=0}^{\infty} (-1)^n \sqrt{\frac{(2n-1)!!}{(2n)!!}} \tanh^n \eta \left| 2n \right\rangle$.

The Bogoliubov transformation appears not only in the BEC but also in optics. The laser is a coherent state, and one can realize Bogoliubov transformed ones called the "squeezed states." They have been under active research.

2. Superfluid Vortex

(a)

The solution is z-independent, and hence is a function of r and θ only. We use the chain rule
$\begin{pmatrix} \nabla_r \\ \nabla_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \nabla_x \\ \nabla_y \end{pmatrix}$
and find
$\begin{pmatrix} \nabla_x \\ \nabla_y \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nabla_r \\ \nabla_\theta \end{pmatrix}$.
Therefore,
$\begin{pmatrix} \nabla_x \\ \nabla_y \end{pmatrix} \psi = \begin{pmatrix} \nabla_x \\ \nabla_y \end{pmatrix} f(r) e^{i n \theta} = \frac{1}{r} \begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} f'(r) \\ i n f(r) \end{pmatrix} e^{i n \theta}$.
The momentum density is
$\vec{j} = \frac{\hbar}{2 m i} f(r) \frac{1}{r} \begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} f'(r) \\ i n f(r) \end{pmatrix} + c.c.$

$\quad = \frac{n\hbar}{m r} f(r)^2 \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$.
On the other hand, the number density is
$\rho = \psi^* \psi = f(r)^2$.
The velocity field is their ratio,
$\vec{v} = \frac{\vec{j}}{\rho} = \frac{n\hbar}{m r} \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$.

For the plotting purpose, we take $\hbar = m = 1$.

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In[7]:= << Graphics`PlotField`
```

$n = 1$

In[8]:= **PlotVectorField**$\left[\frac{1}{\sqrt{x^2+y^2}}\{-y, x\}, \{x, -1, 1\}, \{y, -1, 1\}, \text{ScaleFactor} \to \text{None}, \text{ScaleFunction} \to (.05\ \#\ \&)\right]$

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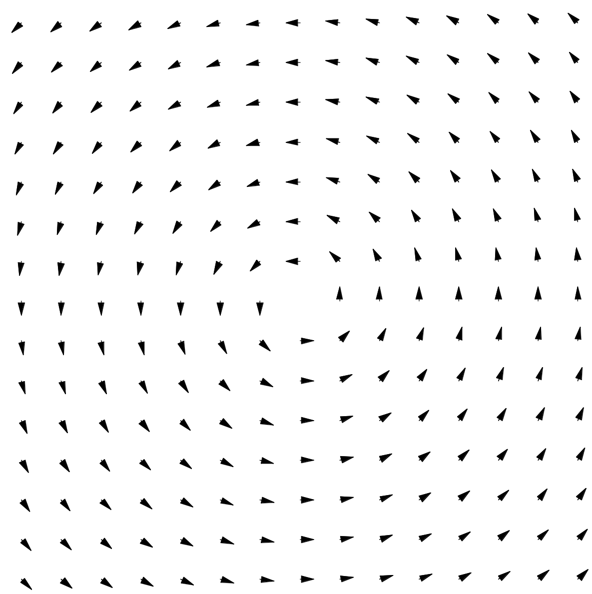
Power::infy : Infinite expression 1/0 encountered. More…
∞::indet : Indeterminate expression 0 ComplexInfinity encountered. More…
∞::indet : Indeterminate expression 0 ComplexInfinity encountered. More…
```



Out[8]= - Graphics -

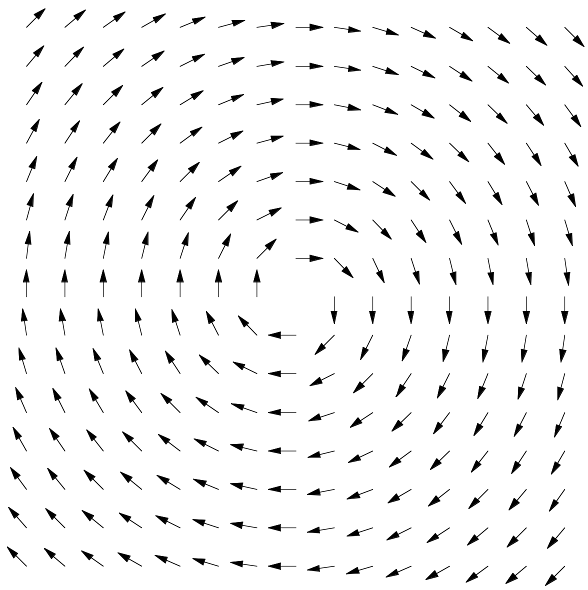
$n = -2$

In[9]:= **PlotVectorField**$\left[\frac{-2}{\sqrt{x^2+y^2}}\{-y, x\}, \{x, -1, 1\}, \{y, -1, 1\}, \text{ScaleFactor} \to \text{None}, \text{ScaleFunction} \to (.05\,\#\,\&)\right]$

Power::infy : Infinite expression $\frac{1}{0}$ encountered. More…

∞::indet : Indeterminate expression 0 ComplexInfinity encountered. More…

∞::indet : Indeterminate expression 0 ComplexInfinity encountered. More…



Out[9]= - Graphics -

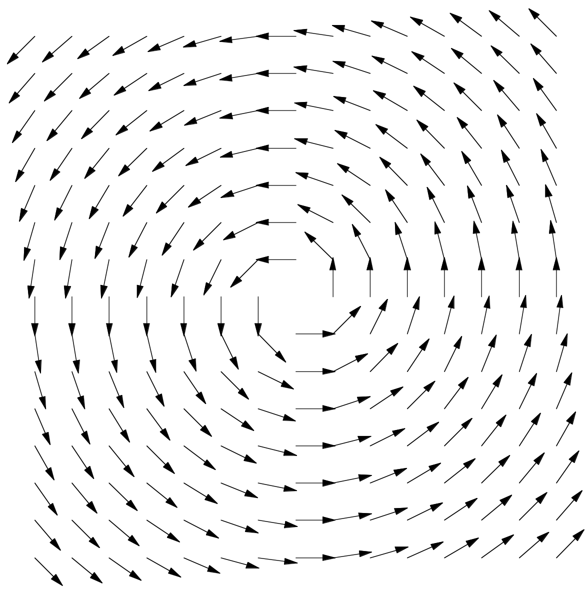
$n = 3$

```
In[10]:= PlotVectorField[3/Sqrt[x^2 + y^2] {-y, x}, {x, -1, 1},
  {y, -1, 1}, ScaleFactor → None, ScaleFunction → (.05 # &)]
```

Power::infy : Infinite expression $\frac{1}{0}$ encountered. More…

∞::indet : Indeterminate expression 0 ComplexInfinity encountered. More…

∞::indet : Indeterminate expression 0 ComplexInfinity encountered. More…



Out[10]= - Graphics -

(b)

The circulation is obtained from the loop integral of the velocity field

$\vec{v} = \frac{\vec{j}}{\rho} = \frac{n\hbar}{m r} \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$,

$\kappa = \oint \frac{n\hbar}{m r} \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \cdot r \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} d\theta$

$= \frac{2\pi n\hbar}{m} = \frac{n h}{m}$

(c)

We start with the Euler-Lagrange equation obtained from the Lagrangian,

$$i\hbar\dot{\psi} + \frac{\hbar^2 \Delta}{2m}\psi + \mu\psi - \lambda\psi^*\psi\psi = 0.$$

We now subsitute our Ansatz into the equation: $\psi(r, \theta, t) = e^{i n \theta} f(r)$.

Because it is static $\dot{\psi} = 0$, we find

$$\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right) f + \mu f - \lambda f^3 = 0.$$

We now try to cast it to a form as independent of the parameters as possible.

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right) f + \frac{2m\mu}{\hbar^2} f - \frac{2m\lambda}{\hbar^2} f^3 = 0$$

We introduce the variable

$$\frac{\sqrt{2m\mu}}{\hbar} r = \rho$$

Then the equation becomes

$$\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{n^2}{\rho^2}\right) f + f - \frac{\lambda}{\mu} f^3 = 0$$

Finally, we introduce the function

$$f = \sqrt{\frac{\mu}{\lambda}}\,\phi$$

which satisfies

$$\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{n^2}{\rho^2}\right)\phi + \phi - \phi^3 = 0$$

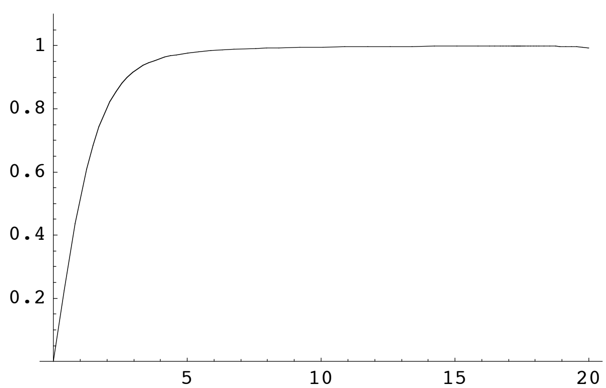
The equation is now completely free from physical parameters except for the integer *n*.

We solve numerically with the boundary conditions $\phi(0) = 0$, $\phi(\infty) = 1$. Instead of imposing the boundary condition stritcly at the origin, which causes all kinds of problems,we cheat a little bit and impose it slightly off the origin.

```
vortex = NDSolve[{y''[x] + 1/x y'[x] - 1/x^2 y[x] + y[x] - y[x]^3 == 0,
   y[10^-6] == 0, y'[10^-6] == 1.166380374636}, y, {x, 10^-6, 20}]
```

```
{{y → InterpolatingFunction[{{1.×10^-6, 20.}}, <>]}}
```

```
Plot[y[x] /. vortex[[1]], {x, 10^-6, 20}, PlotRange → {0, 1.1}]
```

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- Graphics -
```

This way, we see that the vortex solution exists for the classical Schrödinger field with a repulsive self-interaction potential, as observed in the atomic BEC as well as liquid He-II.