221A Lecture Notes
Landau Levels

1 Classical Mechanics

We are interested in a charged particle (electron) in a uniform magnetic field. To simplify the setup, let us consider the electron to move only on xy plane under a magnetic field pointing along the z direction. You can always bring back the motion along the z direction, which is just a translational motion with a constant momentum or velocity. Our interst here is the dynamics on the xy plane. (We use the Gaussian unit in this notes.)

In classical mechanics, all we care is the equation of motion. The Lorentz force causes the electron to spiral around (Larmor motion or cyclotron motion). The centrifugal force must balance the Lorentz force,

$$\frac{mv^2}{r} = \frac{|e|}{c} v|B|, \tag{1.1}$$

and hence

$$r = \frac{mcv}{|eB|}, \tag{1.2}$$

which is the Larmor radius or cyclotron radius. The angular frequency of the cyclotron motion is

$$\omega = 2\pi \frac{v}{2\pi r} = \frac{|eB|}{mc} \tag{1.3}$$

does not depend on the cyclotron radius, and gives a characteristic time-scale for the problem. The energy is given simply by

$$E = \frac{m}{2} v^2 = \frac{m}{2} r^2 \omega^2, \tag{1.4}$$

bigger for larger cyclotron radii. For definiteness, we assume $eB > 0$, namely the magnetic field pointing downwards along the z-axis $B < 0$ for an electron $e < 0$. Then the cyclotron motion is clockwise.

If you use the canonical formalism, you need to specify the vector potential even in the classical mechanics. Even though the system is both translationally and rotationally (around the z-axis) invariant, it is curious that

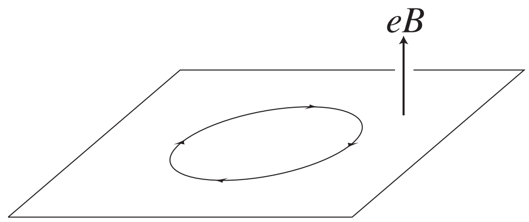

Figure 1: The classical cyclotron or Larmor motion of an electrically charged particle in a uniform mangnetic field.

you cannot find a vector potential that is invariant under both of them. One common choice preserves the rotational invariance (the symmetric gauge),

$$\vec{A} = (A_x, A_y) = \frac{B}{2}(-y, x), \tag{1.5}$$

while the other preserves the translational invariance along the x-axis

$$\vec{A} = (A_x, A_y) = B(-y, 0), \tag{1.6}$$

and yet another preserves that along the y-axis

$$\vec{A} = (A_x, A_y) = B(0, x). \tag{1.7}$$

The Hamiltonian is as we discussed before,

$$H = \frac{1}{2m}\vec{\Pi}^2 = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2. \tag{1.8}$$

The Hamilton equation of motion is then

$$\dot{\vec{x}} = \frac{\partial H}{\partial \vec{p}} = \frac{1}{m}\left(\vec{p} - \frac{e}{c}\vec{A}\right), \tag{1.9}$$

$$\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{x}} = \frac{e}{mc}\left(p_i - \frac{e}{c}A_i\right)\vec{\nabla}A_i. \tag{1.10}$$

Therefore,

$$\begin{aligned}
\ddot{\vec{x}} &= \frac{1}{m}\left(\dot{\vec{p}} - \frac{e}{c}\dot{x}_i\nabla_i\vec{A}\right) \\
&= \frac{1}{m}\left(\frac{e}{mc}\left(p_i - \frac{e}{c}A_i\right)\vec{\nabla}A_i - \frac{e}{c}\frac{1}{m}\left(p_i - \frac{e}{c}A_i\right)\nabla_i\vec{A}\right) \\
&= \frac{e}{mc}\left(v_i\vec{\nabla}A_i - v_i\nabla_i\vec{A}\right) \\
&= \frac{e}{mc}(\vec{v} \times \vec{B}).
\end{aligned} \tag{1.11}$$

This is nothing but the Newton's equation with the Lorentz force $m\ddot{\vec{x}} = \frac{e}{c}(\vec{v} \times \vec{B})$. In components,

$$\ddot{x} = \frac{eB}{mc} v_y = \omega v_y, \tag{1.12}$$

$$\ddot{y} = -\frac{eB}{mc} v_x = -\omega v_x. \tag{1.13}$$

Therefore, a general solution is

$$x = X + r \cos \omega t, \tag{1.14}$$

$$y = Y - r \sin \omega t. \tag{1.15}$$

The energy of the particle is $E = \frac{m}{2}\vec{v}^2$, independent of the magnetic field if expressed in terms of the velocity. Therefore the lowest energy solution is the particle at rest.

The center of the cyclotron motion (X, Y) can be written as

$$\begin{aligned} X &= x + \frac{1}{\omega} v_y, \\ Y &= y - \frac{1}{\omega} v_x. \end{aligned} \tag{1.16}$$

It is easy to verify that they are integrals of motion, namely $\frac{d}{dt}X = \frac{d}{dt}Y = 0$.

If you apply a constant electric field in addition to the magnetic field, the equation of motion is

$$\ddot{\vec{x}} = \frac{e}{mc}(\vec{v} \times \vec{B}) + \frac{1}{m} e\vec{E}. \tag{1.17}$$

Suppose the electric field is along the x-axis. Then the equation of motion can be written explicitly as

$$\ddot{x} = \omega v_y + \frac{1}{m} e E_x, \tag{1.18}$$

$$\ddot{y} = -\omega v_x. \tag{1.19}$$

The solution is

$$x = X + r \cos \omega t, \tag{1.20}$$

$$y = Y - r \sin \omega t - \frac{eE_x}{m\omega} t. \tag{1.21}$$

There is an electric current along the y direction, perpendicular to the direction of the electric field. This current is called the Hall current. It does not depend on the Larmor radius. For every particle in the magnetic field, you get the same contribution.

2 Quantum Mechanics (Generalities)

First important point is that two kinetic momenta do not commute,

$$[\Pi_x, \Pi_y] = [p_x - \frac{e}{c}A_x, p_y - \frac{e}{c}A_y] = i\frac{e\hbar}{c}(\nabla_x A_y - \nabla_y A_x) = i\frac{e\hbar}{c}B_z. \tag{2.1}$$

Because we are interested in a constant $B_z = B$, they have a constant commutator. Another important point is that the canonical momentum and kinetic momentum are different. This point is often a cause of confusions.

We define the z-axis such that $eB > 0$. For the case of the electron $e < 0$, it means that the magnetic field is along the negative z-axis. Then the commutation relation Eq. (2.1) suggests that Π_x and Π_y play the role of position and momentum operator, respectively. Together with the Hamiltonian Eq. (1.8), the system is basically a harmonic oscillator. We define the creation and annihilation operators by

$$a = \sqrt{\frac{c}{2e\hbar B}}(\Pi_x + i\Pi_y), \tag{2.2}$$

$$a^\dagger = \sqrt{\frac{c}{2e\hbar B}}(\Pi_x - i\Pi_y). \tag{2.3}$$

They satisfy

$$[a, a^\dagger] = \frac{c}{2e\hbar B}[\Pi_x + i\Pi_y, \Pi_x - i\Pi_y] = \frac{c}{2e\hbar B} i\frac{e\hbar B}{c}(-i - i) = 1. \tag{2.4}$$

On the other hand, the Hamiltonian Eq. (1.8) becomes

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right), \tag{2.5}$$

with the classical cyclotron frequency Eq. (1.3). The spectrum is that of a harmonic oscillator

$$E = \hbar\omega\left(N + \frac{1}{2}\right) \tag{2.6}$$

with N a non-negative integer. The energy eigenstates are called Landau levels.

The ground state wave function is obtained by solving $a|0\rangle = 0$ as usual, but there is an important difference. There are infinitely many states that satisfy this equation. We will see this point explicitly by employing different gauges.

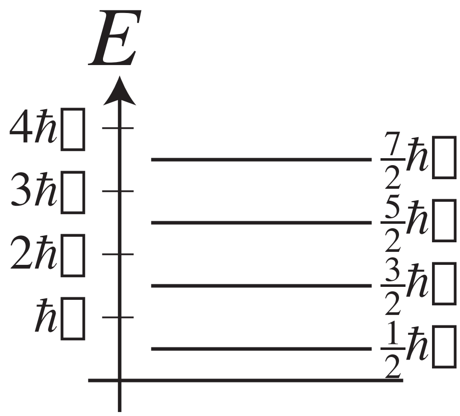

Figure 2: The Landau levels.

Excited states are of course obtained by acting $a^\dagger$ on each ground-state wave function. Again there are infinite number of states at each harmonic oscillator level $N = a^\dagger a$.

The "center of the cyclotron motion" in Eqs. (1.16),

$$X = x + \frac{v_y}{\omega} = x + \frac{c}{eB}\left(p_y - \frac{e}{c}A_y\right), \qquad Y = y - \frac{v_x}{\omega} = y - \frac{c}{eB}\left(p_x - \frac{e}{c}A_x\right). \tag{2.7}$$

commute with Π_x, Π_y, and hence with a, $a^\dagger$, and the Hamiltonian. It is noteworthy that

$$[X, Y] = [x + \frac{\Pi_y}{m\omega}, y - \frac{\Pi_x}{m\omega}] = -\frac{i\hbar}{m\omega} = -\frac{i\hbar c}{eB} \neq 0, \tag{2.8}$$

and hence it is not possible to specify both x- and y-coordinates of the center of the cyclotron motion at the same time. On the other hand, it is possible to introduce another pair or "creation-annihilation operators"

$$b = \sqrt{\frac{eB}{2\hbar c}}(X - iY), \qquad b^\dagger = \sqrt{\frac{eB}{2\hbar c}}(X + iY), \tag{2.9}$$

which satisfy $[b, b^\dagger] = 1$.

3 Rotationally-invariant Gauge

3.1 Ground States

In the gauge Eq. (1.5), the annihilation operator is

$$\begin{aligned} a &= \sqrt{\frac{c}{2e\hbar B}}(\Pi_x + i\Pi_y) \\ &= \sqrt{\frac{c}{2e\hbar B}}\left(p_x - \frac{e}{c}A_x + ip_y - i\frac{e}{c}A_y\right) \\ &= \sqrt{\frac{c}{2e\hbar B}}\left(\frac{\hbar}{i}(\nabla_x + i\nabla_y) - \frac{eB}{2c}(-y+ix)\right) \\ &= \sqrt{\frac{c}{2e\hbar B}}\left(\frac{\hbar}{i}2\frac{\partial}{\partial\bar{z}} - i\frac{eB}{2c}z\right) \\ &= -i\sqrt{\frac{\hbar c}{2eB}}\left(2\frac{\partial}{\partial\bar{z}} + \frac{eB}{2\hbar c}z\right). \end{aligned} \tag{3.1}$$

Here, I introduced the notation $z = x + iy$, $\bar{z} = x - iy$, and

$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}\left(\nabla_x - i\nabla_y\right), \qquad \bar{\partial} = \frac{\partial}{\partial\bar{z}} = \frac{1}{2}\left(\nabla_x + i\nabla_y\right). \tag{3.2}$$

The reason for the factor of a half is to make sure that $\partial z = \bar{\partial}\bar{z} = 1$. Note also $\partial\bar{z} = \bar{\partial}z = 0$. We can hence regard z and $\bar{z}$ as independent variables in partial derivatives.

Similarly, the creation operator in the symmetric gauge is

$$a^\dagger = -i\sqrt{\frac{\hbar c}{2eB}}\left(2\partial - \frac{eB}{2\hbar c}\bar{z}\right). \tag{3.3}$$

It is straightforward to verify $[a, a^\dagger] = 1$.

Solving for the ground state wave function is now easy,

$$\langle z, \bar{z}|a|0\rangle = -i\sqrt{\frac{\hbar c}{2eB}}\left(2\bar{\partial} + \frac{eB}{2\hbar c}z\right)\psi(z,\bar{z}) = 0, \tag{3.4}$$

and we find

$$\psi(z,\bar{z}) = f(z)e^{-eB\bar{z}z/4\hbar c}. \tag{3.5}$$

The prefactor $f(z)$ is an arbitrary function of z. Therefore there are infinitely many ground-state wave functions for each possible analytic function $f(z)$.

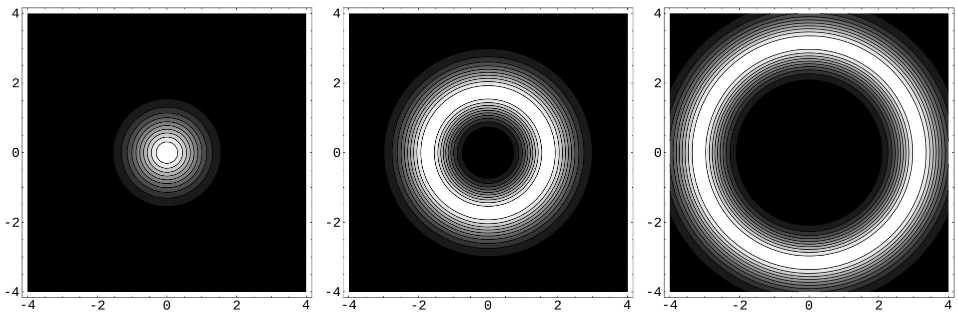

Figure 3: The ground state wave functions with $n = 0$, 3, and 10.

It is customary to choose the independent functions $f(z)$ to be just polynomials z^n:

$$\psi_n(z, \bar{z}) = N_n z^n e^{-eB\bar{z}z/4\hbar c}. \tag{3.6}$$

The overall normalization factor N is fixed by

$$\begin{aligned} \int d^2x |\psi_n|^2 &= N_n^2 \int 2\pi r dr r^{2n} e^{-eBr^2/2\hbar c} \\ &= N_n^2 \pi \int dt\ t^n e^{-eBt/2\hbar c} = N_n^2 \pi \left(\frac{2\hbar c}{eB}\right)^{n+1} \Gamma(n+1) = N_n^2 \pi \left(\frac{2\hbar c}{eB}\right)^{n+1} n!\ . \end{aligned} \tag{3.7}$$

We find

$$N_n = \left(n! \pi \left(\frac{2\hbar c}{eB}\right)^{n+1} \right)^{-1/2} . \tag{3.8}$$

The probability density is basically a ring around the origin. The average radius squared is seen as

$$\langle r^2 \rangle = N_n^2 \pi \int dt\ t^{n+1} e^{-eBt/2\hbar c} = (n+1) \frac{2\hbar c}{eB}, \tag{3.9}$$

further away from the origin for larger n.

Note that n must be non-negative. Otherwise the wave function is not normalizable because of the singularity $z^{-|n|}$ at the origin. Also, only integer powers are allowed because a fractional power would lead to a multi-valued wave function.

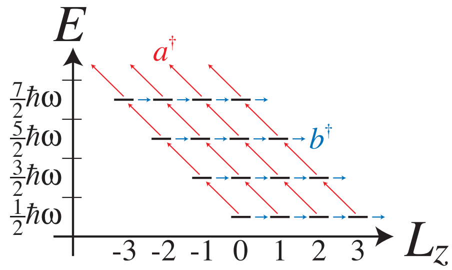

Figure 4: The relationship among various Hamiltonian eigenstates under the action of the creation operators $a^\dagger$ and $b^\dagger$.

The ground states with different n have different eigenvalues of the angular momentum.

$$\begin{aligned} L_z &= xp_y - yp_x \\ &= \frac{\hbar}{i}(x\nabla_y - y\nabla_x) \\ &= \frac{\hbar}{i}\left(\frac{z+\bar{z}}{2}i(\partial - \bar{\partial}) - \frac{z-\bar{z}}{2i}(\partial + \bar{\partial})\right) \\ &= \hbar(z\partial - \bar{z}\bar{\partial}). \end{aligned} \tag{3.10}$$

Therefore, $L_z\psi_n = \hbar n\psi_n$. However, do not think that the state with higher n is rotating faster. The angular momentum is what was obtained with the *canonical* momentum, not the *kinetic* momentum. In fact, L_z is *gauge-dependent* and does not admit direct physical interpretation. The higher Landau levels should be thought of rotating faster. This can be seen from the semi-classical analysis in Section 3.3. It can also be seen by evaluating the expectation value of the *kinetic* angular momentum,

$$\langle x\Pi_y - y\Pi_x\rangle = \langle L_z\rangle - \frac{eB}{2c}\langle r^2\rangle = n\hbar - \frac{eB}{2c}(n+1)\frac{2\hbar c}{eB} = -\hbar, \tag{3.11}$$

which is independent of n for all ground-state wave functions.

The other set of "creation-annihilation operators" $[b, b^\dagger] = 1$ in Eq. (2.9) can be used to relate different ground states. In the complex coordinate,

$$b = \sqrt{\frac{eB}{2\hbar c}}(X - iY)$$

$$
\begin{aligned}
&= \sqrt{\frac{eB}{2\hbar c}} \left(x + \frac{\Pi_y}{m\omega} - iy + i\frac{\Pi_x}{m\omega} \right) \\
&= \sqrt{\frac{\hbar c}{2eB}} \left(2\partial + \frac{eB}{2\hbar c}\bar{z} \right). \qquad (3.12)
\end{aligned}
$$

Similarly,

$$
b^\dagger = \sqrt{\frac{\hbar c}{2eB}} \left(-2\bar{\partial} + \frac{eB}{2\hbar c} z \right). \qquad (3.13)
$$

It is easy to see that they commute with a and $a^\dagger$.

The $n = 0$ ground state is annihilated by b,

$$
b\psi_0 = \sqrt{\frac{\hbar c}{2eB}} \left(2\partial + \frac{eB}{2\hbar c}\bar{z} \right) N_0 e^{-eBz\bar{z}/4\hbar c} = 0. \qquad (3.14)
$$

Other ground states are obtained by acting $b^\dagger$ successively,

$$
\begin{aligned}
(b^\dagger)^n \psi_0 &= \left(\frac{\hbar c}{2eB} \right)^{n/2} \left(\frac{eB}{\hbar c} z \right)^n N_0 e^{-eBz\bar{z}/4\hbar c} \\
&= \left(\frac{eB}{2\hbar c} \right)^{n/2} z^n N_0 e^{-eBz\bar{z}/4\hbar c} = \sqrt{n!}\, N_n z^n e^{-eBz\bar{z}/4\hbar c} \qquad (3.15)
\end{aligned}
$$

with the correct normalization expected from the harmonic oscillator case. It is also easy to see

$$
[L_z, b] = -\hbar b, \qquad [L_z, b^\dagger] = \hbar b^\dagger. \qquad (3.16)
$$

It is consistent with the fact that acting $b^\dagger$ increases the eigenvalue of L_z by one.

3.2 Degeneracy of Ground States

Now we would like to figure out how many states there are, assuming that the system has an overall finite size. Of course, if you discuss a system of finite size, you have to specify the boundary condition. For example, to preserve the rotational invariance of the system, I can impose that all wave functions should vanish at a fixed radius $r = R$. But this will make the discussion quite cumbersome, because we have to talk about Bessel functions and so on. Here, we instead try to answer the question without imposing an explicit boundary condition, but by requiring that states all appear “pretty much” within the radius R. This is definitely not a rigorous way to do it, but if the radius R is large $R \gg \sqrt{\hbar/m\omega}$, there are many many states, while such a rough treatment gives correct result ignoring corrections of $O(1)$. In other words, I don't care if you have 98,375,024 states or 98,375,025 states. Then the job is to figure out within what radius states ψ_n are “pretty much” contained.

What is the most probable value for the radius r for the states ψ_n? This can be answered just by differentiating the probability density $dP = 2\pi r|\psi_n|^2 dr$,

$$\begin{aligned} \frac{d}{dr} 2\pi r|\psi_n|^2 &= 2\pi N_n^2 \frac{d}{dr} r^{2n+1} e^{-\frac{eB}{2\hbar c}r^2} \\ &= 2\pi N_n^2 \left((2n+1) r^{2n} - \frac{eB}{2\hbar c} 2r^{2n+2} \right) e^{-\frac{eB}{2\hbar c}r^2} = 0. \end{aligned} \tag{3.17}$$

We find

$$r_{\max}^2 = \frac{2n+1}{2} \frac{2\hbar c}{eB} = (2n+1) \frac{\hbar c}{eB}. \tag{3.18}$$

The peak radius grows as $n^{1/2}$ for higher n. Beyond this peak radius, the wave function damps very quickly over the distance scale $\sqrt{\frac{2\hbar c}{eB}}$, *i.e.* about where the next wave function ψ_{n+1} starts becoming sizable. Basically, ψ_n has a ring-shaped distribution, and neighboring values of n give you neighboring rings. Assuming the radius R of the system is much larger than the characteristic distance scale for the wave function to damp, $\sim \sqrt{\hbar/m\omega}$, exactly where you draw the line gives you only $O(1)$ correction to counting the number of states. Therefore, we can require that the $r_{\max}^2$ is less than R^2, or

$$(2n+1) \frac{\hbar c}{eB} < R^2, \tag{3.19}$$

and find (further rounding $2n+1$ to $2n$)

$$n \leq \frac{eBR^2}{2\hbar c} = \frac{eB\pi R^2}{2\pi\hbar c} = \frac{e\Phi}{hc}. \tag{3.20}$$

The number of ground states is given by the total magnetic flux going through the system $\Phi = B\pi R^2$ times e/hc.

3.3 Semi-classical Considerations

ψ_n are eigenstates of $L_z = n\hbar$. This can be readily seen by writing $z^n = r^n e^{in\phi}$ using the polar coordinates. But one should not think of L_z as mvr of classical cyclotron motion. Here is why.

When N is large, we expect semi-classical arguments should work. If you require that $\oint \vec{p}\cdot d\vec{r} = 2\pi mvr = (N+\frac{1}{2})h$ over a period of Larmor motion, as we did in the WKB approximation, together with Eq. (1.3), you find

$$E = \frac{1}{2}mv^2 = \frac{1}{2}mvr \times \frac{v}{r} = \frac{1}{2}\left(N+\frac{1}{2}\right)\hbar\omega, \tag{3.21}$$

which disagrees with the true energy levels. This is not because Bohr–Sommerfeld quantization condition fails. Bohr–Sommerfeld quantization condition was justified from the WKB analysis up to an uncertainty of $O(h)$ compared to Nh. The apparent discrepancy is due to the fact that p is *different* from the kinetic momentum, and we have to take care of the difference.[1] The true condition is

$$\begin{aligned}\left(N+\frac{1}{2}\right)h &= \oint \vec{p}\cdot d\vec{r} = \oint \left(m\vec{v} + \frac{e}{c}\vec{A}\right)\cdot d\vec{r} \\ &= 2\pi mvr - \frac{e}{c}B\pi r^2 = 2\pi mvr - \pi m\omega r^2. \end{aligned} \tag{3.22}$$

(Note that the Larmor motion is clockwise for $eB > 0$ we have been assuming. That is why the line integral of the vector potential gives the negative of the magnetic flux inside the Larmor radius.) Multiplying both sides with $\omega/2\pi$, and using $\omega = v/r$, we find

$$\left(N+\frac{1}{2}\right)\hbar\omega = \frac{1}{2}mv^2 = E, \tag{3.23}$$

which agrees exactly with the Landau levels we had obtained, including due to the "zero point energy." (Just like in the case of the harmonic oscillator, this exact agreement must be considered a luck or coincidence.) The vector potential is crucial in this comparison.

Given the above considerations, each Landau level can be thought of a quantized cyclotron motion with $mvr = \hbar(2N+1)$, even though each state may have a different *canonical* angular momentum $L_z = n\hbar$. This point can

[1] I thank Paul McEuen for this observation.

indeed be seen by working out the probability current density for the ground state wave function with $L_z = n\hbar$,

$$\begin{aligned}\vec{\jmath} &= \frac{1}{2m}\left(\psi_n^*\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}\right)\psi_n + \left(-\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}\right)\psi_n^*\psi_n\right) \\ &= \frac{1}{2m}\left[-2\hbar n + \frac{eB}{c}r^2\right](\bar{z}z)^{n-1}e^{-\frac{eB}{2\hbar c}\bar{z}z}(y,-x). \qquad (3.24)\end{aligned}$$

At the most probable radius $r^2 = (2n+1)\frac{\hbar c}{eB}$, the terms in the square bracket nearly cancel and give a contribution of about $-\hbar$, independent of n. Therefore, even though the ground states at higher n have larger radii and higher angular momenta, they do not rotate any faster; the probability current for these sates can be viewed as that of the "zero-point motion".

It is useful to note that indeed a classical particle at rest can carry an angular momentum. In this gauge, the angular momentum is

$$L_z = \vec{r}\times\vec{p} = \vec{r}\times\left(m\vec{v} + \frac{e}{c}\vec{A}\right) = m\vec{r}\times\vec{v} + \frac{eB}{2c}r^2. \qquad (3.25)$$

If you place the electron at rest at the radius r away from the origin, it has the angular momentum $\frac{eB}{2c}r^2$. Because $r^2 = 2n\frac{\hbar c}{eB}$ semi-classically, we find $L_z = n\hbar$ indeed. This observation demonstrates the fact that the angular momentum L_z does not correspond to the classical cyclotron motion.

3.4 Excited States

We can act the creation operator Eq. (3.3) on the general ground state,

$$\begin{aligned}a^\dagger\psi_n &= -i\sqrt{\frac{\hbar c}{2eB}}\left(2\partial - \frac{eB}{2\hbar c}\bar{z}\right)N_n z^n e^{-eBz\bar{z}/4\hbar c} \\ &= -i\sqrt{\frac{\hbar c}{2eB}}\left(2nz^{n-1} - 2\frac{eB}{4\hbar c}\bar{z} - \frac{eB}{2\hbar c}\bar{z}\right)N_n z^n e^{-eBz\bar{z}/4\hbar c} \\ &= -i\sqrt{\frac{\hbar c}{2eB}}\left(2n - \frac{eB}{\hbar c}\bar{z}z\right)N_n z^{n-1} e^{-eBz\bar{z}/4\hbar c}. \qquad (3.26)\end{aligned}$$

It is easy to see that it has the energy $H = \frac{3}{2}\hbar\omega$, and the angular momentum $L_z = (n-1)\hbar$. On the other hand, the *kinetic* angular momentum can be worked out similarly to the ground state, by noting $\langle r^2\rangle = \frac{2\hbar c}{eB}(n+2)$. We find $\langle x\Pi_y - y\Pi_x\rangle = -3\hbar$ independent of n, consistent with the semi-classical argument.

By acting the creation operator multiple times, one obtains any excited states. Each Landau level has the same degeneracy.

3.5 Coherent State

The classical Larmor motion is most closely approximated by a coherent state, just like for the harmonic oscillator. The wave function

$$\psi = N_0 e^{-eB(z\bar{z}-2z_0\bar{z}+z_0\bar{z}_0)/4\hbar c}. \tag{3.27}$$

is an eigenstate of the annihilation operator

$$a\psi = -i\sqrt{\frac{\hbar c}{2eB}}\left(-\frac{eB}{2\hbar c}(z-2z_0)+\frac{eB}{2\hbar c}z\right)\psi = -i\sqrt{\frac{eB}{2\hbar c}}z_0. \tag{3.28}$$

Note that the coherent state is *not* a linear combination of just ground-state wavefunctions, but contains excited states. This can be seen by expanding the exponent as

$$\psi = e^{-\frac{eB}{4\hbar c}z_0\bar{z}_0}e^{\frac{eB}{2\hbar c}z_0\bar{z}}\psi_0. \tag{3.29}$$

The first factor is just a constant. However, the second factor depends on $\bar{z}$, and takes the state to higher Landau levels. Therefore, the coherent state is a linear combination of all Landau levels, the same as in the harmonic oscillator case.

This wave function is constructed the same way it is for the harmonic oscillator, $e^{fa^\dagger}|0\rangle e^{-f\bar{f}/2}$. Using the expression for the creation operator in Eq. (3.3),

$$e^{fa^\dagger}\psi_0 e^{-f\bar{f}/2} = N_0 e^{i\sqrt{\frac{eB}{2\hbar c}}\ f\bar{z}}e^{-eBz\bar{z}/4\hbar c}e^{-f\bar{f}/2}. \tag{3.30}$$

Choosing $f = -i\sqrt{\frac{eB}{2\hbar c}}z_0$, it reproduces Eq. (3.27) above.

It is centered at $z = z_0$. To see this, we calculate the probability density

$$\begin{aligned} |\psi|^2 &= N_0^2 \exp\left(-\frac{eB}{4\hbar c}(2z\bar{z}-2z_0\bar{z}-2\bar{z}_0 z+2z_0\bar{z}_0)\right) \\ &= N_0^2 \exp\left(-\frac{eB}{2\hbar c}(z-z_0)(\bar{z}-\bar{z}_0)\right) \\ &= N_0^2 \exp\left[-\frac{eB}{2\hbar c}\left((x-x_0)^2+(y-y_0)^2\right)\right]. \end{aligned} \tag{3.31}$$

Therefore, it is a Gaussian centered at $z = z_0$, or $(x, y) = (x_0, y_0)$ for $z_0 = (x_0 + iy_0)$, and the normalization is given correctly by N_0 in Eq. (3.8).

Using the probability density given above, it is easy to see that

$$\langle x \rangle = x_0, \qquad \langle y \rangle = y_0. \tag{3.32}$$

The variance is also calculated very easily,

$$\langle (x - x_0)^2 \rangle = \langle (y - y_0)^2 \rangle = \frac{\hbar c}{eB}. \tag{3.33}$$

The expectation values of the momentum operators are calculated in the usual way. First we rewrite the wave function in x and y instead of z and $\bar{z}$,

$$\begin{aligned} \psi &= N_0 \exp \frac{-eB}{4\hbar c} \left(x^2 + y^2 - 2(x_0 + iy_0)(x - iy) + (x_0^2 + y_0^2) \right) \\ &= N_0 \exp \frac{-eB}{4\hbar c} \left((x - x_0)^2 + (y - y_0)^2 - 2(-ix_0 y + iy_0 x) \right). \end{aligned} \tag{3.34}$$

Therefore,

$$\langle p_x \rangle = \int dxdy |\psi|^2 \frac{-eB}{4\hbar c} \frac{\hbar}{i} \left(2(x - x_0) - 2iy_0 \right) = \frac{eB}{2c} y_0, \tag{3.35}$$

and

$$\langle p_y \rangle = \int dxdy |\psi|^2 \frac{-eB}{4\hbar c} \frac{\hbar}{i} \left(2(y - y_0) + 2ix_0 \right) = -\frac{eB}{2c} x_0. \tag{3.36}$$

Now we proceed to the variance.

$$\begin{aligned} \langle p_x^2 \rangle &= \int dxdy \left| \frac{-eB}{4\hbar c} \frac{\hbar}{i} \left(2(x - x_0) - 2iy_0 \right) \psi \right|^2 \\ &= \int dxdy |\psi|^2 \frac{e^2 B^2}{4c^2} \left((x - x_0)^2 + y_0^2 \right) \\ &= \frac{e^2 B^2}{4c^2} \frac{\hbar c}{eB} + \frac{e^2 B^2}{4c^2} y_0^2, \end{aligned} \tag{3.37}$$

and hence

$$(\Delta p_x)^2 = \langle p_x^2 \rangle - \langle p_x \rangle^2 = \frac{e\hbar B}{4c}. \tag{3.38}$$

The result for $(\Delta p_y)^2$ is the same. The uncertainty relation is therefore

$$(\Delta x)^2 (\Delta p_x)^2 = \frac{\hbar c}{eB} \frac{e\hbar B}{4c} = \frac{\hbar^2}{4}. \tag{3.39}$$

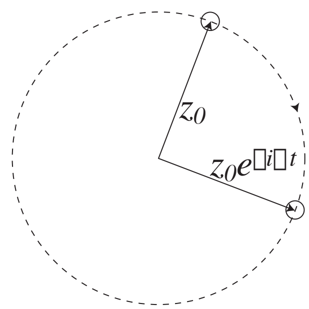

Figure 5: The time evolution of the coherent state Eq. (3.27).

It is the same for $(\Delta y)^2(\Delta p_y)^2$. This state has the minimum uncertainty.

Time evolution of this state is obtained again the same way as in the harmonic oscillator,

$$\begin{aligned} e^{-iHt/\hbar}e^{fa^\dagger}|0\rangle &= e^{-iHt/\hbar}e^{fa^\dagger}e^{iHt/\hbar}e^{-iHt/\hbar}|0\rangle \\ &= e^{fa^\dagger e^{-i\omega t}}|0\rangle e^{-i\omega t/2}. \end{aligned} \tag{3.40}$$

Therefore, apart from the overall phase factor $e^{-i\omega t/2}$ due to the zero-point energy, the time evolution is given by replacing the center of the wave function z_0 with $z_0e^{-i\omega t}$. Namely,

$$\psi(t) = N_0 e^{-eB(z\bar{z}-2z_0\bar{z}e^{-i\omega t}+z_0\bar{z}_0)}e^{-i\omega t/2}. \tag{3.41}$$

One can verify explicitly that it satisfies the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}\psi(t) = H\psi(t) = \hbar\omega\left(\frac{eB}{2\hbar c}z_0e^{-i\omega t}\bar{z}+\frac{1}{2}\right)\psi(t). \tag{3.42}$$

The probability density is

$$\begin{aligned} |\psi(t)|^2 &= N_0^2 e^{-eB(2z\bar{z}-2z_0\bar{z}e^{-i\omega t}-2z\bar{z}_0e^{i\omega t}+2z_0\bar{z}_0)/4\hbar c} \\ &= N_0^2 e^{-eB(z-z_0e^{-i\omega t})(\bar{z}-\bar{z}_0e^{i\omega t})/2\hbar c}. \end{aligned} \tag{3.43}$$

Therefore it has the same Gaussian shape at all times centered at $z_0e^{-i\omega t}$. Note that the time-dependence of the position $z_0(t) = z_0e^{-i\omega t}$ is nothing but an overall cyclotron motion of the particle. The motion is clockwise, as expected from the purely classical intuition.

Another interesting one is a manifestly Gaussian wave function centered at $z = z_0$,

$$\phi = N_0 e^{-eB(z-z_0)(\bar{z}-\bar{z}_0)/4\hbar c}. \tag{3.44}$$

This state is also an eigenstate of the annihilation operator,

$$\begin{aligned} a\phi &= -i\sqrt{\frac{\hbar c}{2eB}}\left(2\bar{\partial}+\frac{eB}{2\hbar c}z\right)\phi \\ &= -i\sqrt{\frac{\hbar c}{2eB}}\left(2\frac{-eB}{4\hbar c}(z-z_0)+\frac{eB}{2\hbar c}z\right)\phi \\ &= -\frac{i}{2}\sqrt{\frac{eB}{2\hbar c}}z_0\phi, \end{aligned} \tag{3.45}$$

with a half the eigenvalue of Eq. (3.27). This state can be obtained from the off-centered ground-state wave function at $z = z_0/2$ (see the next section),

$$N_0 e^{-eB(z\bar{z}-z\bar{z}_0+z_0\bar{z}_0/4)/4\hbar c}, \tag{3.46}$$

and make it into a coherent state with the operator

$$e^{-i\sqrt{\frac{2eB}{\hbar c}}\ \frac{z_0}{4}a^\dagger}e^{-eBz_0\bar{z}_0/16\hbar c} = e^{-\frac{z_0}{4}(2\partial-\frac{eB}{2\hbar c}\bar{z})}e^{-eBz_0\bar{z}_0/16\hbar c}. \tag{3.47}$$

Because $2\partial = -\frac{eB}{2\hbar c}(\bar{z}-\bar{z}_0)$ on the above off-centered state, acting this operator yields

$$\begin{aligned} & e^{-\frac{z_0}{4}(2\partial-\frac{eB}{2\hbar c}\bar{z})}e^{-eBz_0\bar{z}_0/16\hbar c}N_0 e^{-eB(z\bar{z}-z\bar{z}_0+z_0\bar{z}_0/4)/4\hbar c} \\ &= e^{-\frac{z_0}{4}(-\frac{eB}{2\hbar c}(\bar{z}-\bar{z}_0)-\frac{eB}{2\hbar c}\bar{z})}e^{-eBz_0\bar{z}_0/16\hbar c}N_0 e^{-eB(z\bar{z}-z\bar{z}_0+z_0\bar{z}_0/4)/4\hbar c} \\ &= N_0 e^{-eB(z-z_0)(\bar{z}-\bar{z}_0)/4\hbar c}. \end{aligned} \tag{3.48}$$

The time evolution of this state can be worked out in the same way as the previous example, and we find

$$\phi(t) = N_0 \exp\left[-\frac{eB}{4\hbar c}\left(z\bar{z}-z\bar{z}_0-z_0e^{-i\omega t}\bar{z}+z_0\frac{1+e^{-i\omega t}}{2}\bar{z}_0\right)\right]e^{-i\omega t/2}. \tag{3.49}$$

It shows a Gaussian shape centered at $z = z_0\frac{1+e^{-i\omega t}}{2}$, namely a cyclotron motion around $z_0/2$ going through z_0 and the origin. One can check explicitly that it satisfies the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}\phi(t) = H\phi(t) = \hbar\omega\left[\frac{eB}{4\hbar c}z_0e^{-i\omega t}\left(\bar{z}-\frac{\bar{z}_0}{2}\right)+\frac{1}{2}\right]\phi(t). \tag{3.50}$$

3.6 Translational Invariance

The gauge Eq. (1.5) is not translationally invariant. However, the system is clearly translationally invariant and we expect there is a conserved quantity due to Noether's theorem.

The tricky point is that the spatial translation changes the vector potential Eq. (1.5), but an additional gauge transformation can bring it back to the same form. Under the spatial translation $(x, y) \to (x + a, y + b)$,

$$\vec{A} = (A_x, A_y) = \frac{B}{2}(-y, x) \to \frac{B}{2}(-y - b, x + a) = \frac{B}{2}(-y, x) + \vec{\nabla}(ay - bx). \tag{3.51}$$

Correspondingly, the Lagrangian changes by a total derivative

$$\begin{aligned} \delta L &= \delta\left(-\frac{e}{c}\vec{A}\cdot\dot{\vec{x}}\right) = -\frac{eB}{2c}(-\delta y \dot{x} + \delta x \dot{y}) \\ &= -\frac{eB}{2c}(-b\dot{x} + a\dot{y}) = -\frac{d}{dt}\frac{eB}{2c}(ay - bx). \end{aligned} \tag{3.52}$$

In general, if an infinitemesimal change in the variables keeps the Lagrangian the same up to a total derivative $\frac{d}{dt}K$, the conserved quantity can be obtained as follows.

$$\begin{aligned} \delta \int_{t_i}^{t_f} dt L &= \int_{t_i}^{t_f} dt \left(\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x} \right) \\ &= \left. \frac{\partial L}{\partial \dot{x}}\delta x \right|_{t_i}^{t_f} + \int_{t_i}^{t_f} dt \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right) \delta x. \end{aligned} \tag{3.53}$$

The second term vanishes because of the Euler–Lagrange equation, while the the change in the action is (by definition)

$$\delta \int_{t_i}^{t_f} dt L = \int_{t_i}^{t_f} dt \frac{d}{dt} K = K(t_f) - K(t_i). \tag{3.54}$$

Setting Eqs. (3.53,3.54) the same, we find the quantity

$$\frac{\partial L}{\partial \dot{x}}\delta x - K \tag{3.55}$$

is conserved.

In our case, the same argument tells us that

$$p_x - \frac{eB}{2c}y, \qquad p_y + \frac{eB}{2c}x \tag{3.56}$$

are conserved. It is easy to check that they commute with

$$\Pi_x = p_x + \frac{eB}{2c}y, \qquad \Pi_y = p_y - \frac{eB}{2c}x \tag{3.57}$$

and hence also with the Hamiltonian $H = \frac{1}{2m}(\Pi_x^2 + \Pi_y^2)$.

By applying a translation generated by $p_x - \frac{eB}{2c}y$, we find that the state ψ_n is transformed to

$$e^{i(p_x - \frac{eB}{2c}y)a/\hbar} N_n z^n e^{-\frac{eB}{4\hbar c}\bar{z}z} = N_n (z+a)^n e^{-\frac{eB}{2\hbar c}za} e^{-\frac{eB}{4\hbar c}a^2} e^{-\frac{eB}{4\hbar c}\bar{z}z}. \tag{3.58}$$

Because the prefactor depends only on z but not on $\bar{z}$, this is indeed another ground state wave function.

Note that the translation operators that commute with the Hamiltonian are, up to the normalization, nothing but the "center of the cyclotron motion" in Eqs. (1.16),

$$X = x - \frac{v_y}{\omega} = x - \frac{c}{eB}\left(p_y - \frac{e}{c}A_y\right), \qquad Y = y + \frac{v_x}{\omega} = y + \frac{c}{eB}\left(p_x - \frac{e}{c}A_x\right). \tag{3.59}$$

More generally, we can use the linear combinations of X and Y in the form of b and $b^\dagger$ in Eqs. (2.9,3.12,3.13). Because $b\psi_0 = 0$ (Eq. (3.14), only $b^\dagger$ is useful on ψ_0, and hence we are led to consider "coherent state"

$$e^{-\sqrt{\frac{eB}{2\hbar c}}\bar{z}_0 b^\dagger} \psi_0 e^{-eBz_0\bar{z}_0/4\hbar c} = N_0 e^{-eB(z\bar{z} - 2z\bar{z}_0 + z_0\bar{z}_0)/4\hbar c}. \tag{3.60}$$

It looks deceivingly similar to the true coherent state wave function Eq. (3.27), but notice the difference between $z\bar{z}_0$ vs $z_0\bar{z}$. This one is a ground-state wave function, while the true coherent state has the excited states.

3.7 Laughlin's Wave Function

The fact that all wavefunctions at the lowest Landau level are given by a factor z^n played a crucial role in Laughlin's theory of fractional quantum Hall effect, for which he was awarded Nobel prize. If you fill electrons in all of the lowest Landau level, one can show that appropriately anti-symmeterized multi-electron wave function is

$$\Psi = \prod_{i<j} (z_i - z_j) e^{-\frac{eB}{4\hbar c} \sum_i \bar{z}_i z_i}. \tag{3.61}$$

You can see that the highest possible power in one of the coordinates, say z_1, is $N - 1$, where N is the number of particles. (We again ignore the difference between $N - 1$ and N.) If you fill one electron to each of the lowest Landau level, the number of particles is $N = \frac{e\Phi}{hc}$, which is indeed the highest power of z allowed. What Laughlin did to describe a fractional filling $1/k$ (k is always odd) with a surprising stability as observed by D.C. Tsui, H.L. Stormer, A.C. Gossard, *Phys. Rev. Lett.* **48**, 1559–1562(1982); `http://prola.aps.org/abstract/PRL/v48/i22/p1559_1` was to write the wave function (R.B. Laughlin, *Phys. Rev. Lett.* **50**, 1395–1398 (1983); `http://link.aps.org/abstract/PRL/v50/p1395`)

$$\Psi = \prod_{i<j} (z_i - z_j)^k e^{-\frac{eB}{4\hbar c} \sum_i \bar{z}_i z_i}. \tag{3.62}$$

Then the highest power of z is $k(N - 1)$, and hence you can put in only $N = \frac{1}{k} \frac{e\Phi}{hc}$ particles. Indeed, this wave function describes a fractional filling of filling factor $1/k$.

A surprising prediction of Laughlin's theory was that, if you create a "hole" on the Laughlin state at position z_0 by

$$\Psi = \prod_i (z_i - z_0) \prod_{i<j} (z_i - z_j)^k e^{-\frac{eB}{4\hbar c} \sum_i \bar{z}_i z_i}, \tag{3.63}$$

to move all the electrons away from z_0, you will lose the overall electric charge of $\frac{1}{k} e$. This is because the additional factor *adds* to the power in z_1 by one so that you have to decrease the number of particles in the second factor for the system by $1/k$ to fit in the same radius. The predicted fractionally charged excitation had been observed experimentally. See, *e.g.*, L. Saminadayar, D. C. Glattli, Y. Jin, and B. Etienne, *Phys. Rev. Lett.* **79**, 2526–2529 (1997), `http://prola.aps.org/abstract/PRL/v79/i13/p2526_1`.

4 Translationally-invariant Gauge

Let us pick the gauge Eq. (1.7). This gauge preserves the translational invariance along the y direction. The case with the gauge Eq. (1.6) is comletely analogous.

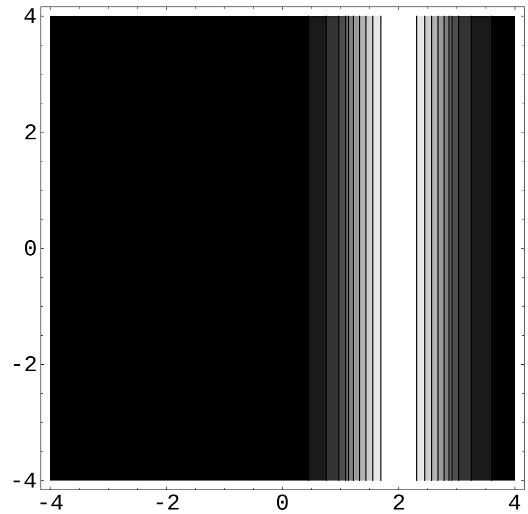

Figure 6: The ground state wave function Eq. (4.4).

The annihilation operator Eq. (2.2) is given in this gauge as

$$a = \sqrt{\frac{c}{2e\hbar B}}\,\frac{\hbar}{i}\left(\nabla_x - \left(-i\nabla_y - \frac{eB}{\hbar c}x\right)\right). \tag{4.1}$$

Note that there is no y dependence as expected from the translational invariance, and hence we can take plan wave solutions

$$\psi(x,y) = \psi(x)e^{ik_y y}. \tag{4.2}$$

Then the annihilation operators becomes

$$a = \sqrt{\frac{c}{2e\hbar B}}\,\frac{\hbar}{i}\left(\nabla_x - \left(k_y - \frac{eB}{\hbar c}x\right)\right). \tag{4.3}$$

The (unnormalized) ground-state wave function is then easily obtained as

$$\psi_0 = \exp\left(ik_y y - \frac{eB}{2\hbar c}\left(x - \frac{\hbar c}{eB}k_y\right)^2\right). \tag{4.4}$$

The wave function is a strip stretched along the y axis, while is centered at $x = \frac{\hbar c}{eB}k_y$ as a Gaussian.

Note that the center of the cyclotron motion Eq. (1.16) in this gauge is given by

$$X = x + \frac{c}{eB}\left(p_y - \frac{eB}{c}x\right) = \frac{c}{eB}p_y, \qquad Y = y - \frac{c}{eB}p_x. \tag{4.5}$$

Therefore, Eq. (4.4) is an eigenstate of $X = \frac{\hbar c}{eB}k_y$, precisely where the strip is located along the x-axis.

One can again ask how many states there are. Let us say that the system is approximately rectangular $d_x \times d_y$ in size. For the center of the wave function to be contained in $[0, d_x]$ along the x-direction, we need $k_y \in [0, \frac{eB}{\hbar c}d_x]$. On the other hand, if we employ a periodic boundary condition along the y axis, k_y is quantized: $k_y = 2\pi n/d_y$. Therefore, within the range allowed for k_y, there are $\frac{eB}{\hbar c}d_x\frac{d_y}{2\pi} = \frac{e\Phi}{hc}$ with the magnetic flux $\Phi = Bd_xd_y$. This is the same counting as in the case of the rotationally-invariant gauge. If the system is large enough, we don't expect the result to depend on details of the boundary conditions nor the shape of the system. The agreement confirms this expectation.

The probability current of the ground-state wave function is

$$\vec{\jmath} = \frac{1}{m}\left(\hbar k_y - \frac{eB}{c}x\right)|\psi_0|^2(0,1). \tag{4.6}$$

Therefore, it is only along the y-direction, but the probability flows along the negative y direction for $x > \frac{\hbar c}{eB}k_y$ while along the positive y direction for $x < \frac{\hbar c}{eB}k_y$. At the center of the wave function, there is no probability flow. Overall, there is no probabiliy flow after integrating over the x direction.

This gauge is particularly useful to study the Hall current because of its translational invariance. A constant electric field along the positive x-direction can be added to the Hamiltonian as

$$H = \frac{1}{2m}(\Pi_x^2 + \Pi_y^2) - eEx = \frac{1}{2m}\left(p_x^2 + \left(p_y - \frac{eB}{c}x\right)^2\right) - eEx. \tag{4.7}$$

Using plane waves along the y direction e^{ik_yy} again,

$$H = \frac{1}{2m}\left(p_x^2 + \left(\hbar k_y - \frac{eB}{c}x\right)^2\right) - eEx. \tag{4.8}$$

We can complete the square for x,

$$H = \frac{1}{2m}\left(p_x^2 + \left(\hbar k_y - \frac{eB}{c}x + \frac{mc}{eB}eE\right)^2\right) - \hbar k_y\frac{c}{eB}eE - \frac{1}{2m}\left(\frac{mc}{eB}eE\right)^2. \tag{4.9}$$

The ground-state wave function peaks at a shifted x,

$$\psi_0 = \exp\left(ik_y y - \frac{eB}{2\hbar c}\left(x - \frac{\hbar c}{eB}k_y - m\left(\frac{c}{eB}\right)^2 eE\right)^2\right). \tag{4.10}$$

The probability current is still given by the same expression Eq. (4.6), but the peak value of x is different. Therefore, there is a net flow of probability along the negative y direction (if $eE > 0$), as expected from classical considerations. At the center $x = \frac{\hbar c}{eB}k_y + m\left(\frac{c}{eB}\right)^2 eE$, the velocity of the probability flow is

$$v_y = \frac{1}{m}\left(\hbar k_y - \frac{eB}{c}x\right) = -\frac{c}{eB}eE = -\frac{eE}{m\omega}. \tag{4.11}$$

This is again what we expect from classical considerations.

5 Spin and Supersymmetry

It is straightforward to include the electron spin into the Hamiltonian with the magnetic moment $\vec{\mu} = g\frac{e\vec{s}}{2mc}$,

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) - g\frac{e}{2mc}\vec{s}\cdot\vec{B} = \hbar\omega\left(a^\dagger a + \frac{1}{2} - \frac{g}{4}\sigma_z\right). \tag{5.1}$$

Here, we used the fact $\omega = \frac{eB}{mc}$ and $s_z = \hbar\sigma_z/2$.

There is a remarkable point about the electron with $g = 2$. The ground states with $\sigma_z = +1$ has precisely zero energy, while the ground states with $\sigma_z = -1$ and the first Landau level with $\sigma_z = +1$ are degenerate.

This point can be understood by defining the operator

$$Q = \sqrt{\frac{1}{2m}}\ (\sigma_x\Pi_x + \sigma_y\Pi_y) \tag{5.2}$$

The square of this operator is

$$\begin{aligned} Q^2 &= \frac{1}{2m}\left(\Pi_x^2 + \Pi_y^2 + \sigma_x\sigma_y\Pi_x\Pi_y + \sigma_y\sigma_x\Pi_y\Pi_x\right) \\ &= \frac{1}{2m}\left(\sigma_x^2\Pi_x^2 + \sigma_y^2\Pi_y^2 + i\sigma_z[\Pi_x,\Pi_y]\right) \\ &= \frac{1}{2m}(\Pi_x^2 + \Pi_y^2) - \frac{e\hbar B}{2mc}\sigma_z = H, \end{aligned} \tag{5.3}$$

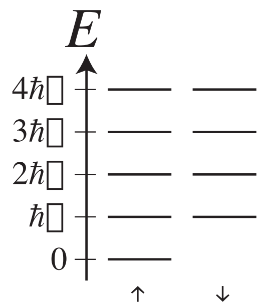

Figure 7: The Landau levels with spin and $g = 2$. Each level is degenerate between the spin up and down states, while the ground state has precisely zero energy with only the spin up state.

precisely the Hamiltonian with $g = 2$. One can further rewrite this operator as a matrix

$$\begin{aligned} Q &= \sqrt{\frac{1}{2m}} \begin{pmatrix} 0 & \Pi_x - i\Pi_y \\ \Pi_x + i\Pi_y & 0 \end{pmatrix} \\ &= \sqrt{\frac{1}{2m}} \sqrt{\frac{2e\hbar B}{c}} \begin{pmatrix} 0 & a^\dagger \\ a & 0 \end{pmatrix} \\ &= \sqrt{\hbar\omega} \begin{pmatrix} 0 & a^\dagger \\ a & 0 \end{pmatrix}. \end{aligned} \tag{5.4}$$

Therefore, Q takes the state to a higher Landau level and raises the spin, or to a lower Landau level and lowers the spin. Because Q commutes with the Hamiltonian, Q does not change the energy. This way, we can see why each Landau level is degenerate between the two spin states. Namely, each state comes in degenerate pairs, $|i\rangle$ and $Q|i\rangle$. However, the ground state, namely the lowest Landau level with spin up

$$\begin{pmatrix} \psi_0 \\ 0 \end{pmatrix}, \tag{5.5}$$

is annihilated by Q, and is not degenerate with the opposite spin state. Moreover, it has precisely zero energy $H = Q^2 = 0$.

This operator Q is called *supercharge*, and the symmetry generated by it *supersymmetry*. In general, when there is a set operators that anti-commute

$\{Q_i, Q_j\} = 0$ for $i \neq j$, and if the Hamiltonian is written as a sum of squares, $H = Q_1^2 + \cdots + Q_n^2$, the operators Q_i are called supercharges. Then the Hamiltonian commutes with all supercharges, and hence the supercharges are conserved. The Hamiltonian is said to be *supersymmetric*. The ground state is annihilated by them $Q_i|0\rangle = 0$ for all i. The pair of states related by the action of a supercharge is called *superpartners*. The relativistic version of supersymmetry is a crucial ingredient of the string theory.

A Precise Counting of the Degeneracy

This appendix is only for the mathematically inclined, and you can skip it entirely if not interested. The degeneracy of the ground states can be studied exactly if the system is not an open plane but rather a compact Riemann surface. We set $e = c = \hbar = 1$ to simplify notation. Then the conclusion from Section 3.2 is that the number of states is approximately $N \simeq \frac{1}{2\pi}\Phi$, where $\Phi = B\pi R^2$ is the total magnetic flux.

Let us reformulate the problem of solving for the ground state wave functions so that it can be generalized to Riemann surfaces. We can regard the electromagnetism with a background magnetic field in two-dimensions as a complex line bundle over a complex plane. The gauge connection in the symmetric gauge is

$$\begin{aligned} A &= \frac{B}{2}(-ydx + xdy) \\ &= \frac{B}{2}\left(\frac{i(z-\bar{z})}{2}\frac{dz+d\bar{z}}{2} + \frac{z+\bar{z}}{2}\frac{-i(dz-d\bar{z})}{2}\right) = \frac{B}{4}(-i\bar{z}dz + izd\bar{z}). \end{aligned} \tag{A.6}$$

Using a "complexified" gauge transformation, *i.e.*, extending the structure group from U(1) to GL(1,$\mathbb{C}$), we can transform the gauge connection to $A' = A + (\partial + \bar{\partial})\Lambda$,

$$A' = -i\frac{B}{2}\bar{z}dz \tag{A.7}$$

with the gauge parameter $\Lambda = -i\frac{B}{4}\bar{z}z$. Namely, $A_z = -i\frac{B}{2}\bar{z}$, $A_{\bar{z}} = 0$. Note that this gauge transformation changes the inner product of the wave functions to $\int dxdy\psi^*\psi e^{-B\bar{z}z/2}$ because GL(1,$\mathbb{C}$) is not unitary. Then the ground state equation in this gauge is simply $(\bar{\partial} + A_{\bar{z}})\psi = \bar{\partial}\psi = 0$. Therefore, the question is to find a complete set of holomorphic functions. As seen below, the precise mathematical question is: *how many holomorphic sections are there for a twisted holomorphic line bundle on a Riemann surface?*

A.1 S^2

As a way of making the system to have a finite area, let us make the plane into a sphere S^2 by identifying the infinity in all directions. The first point to note is that the system is now basically the surface that surrounds a magnetic monopole. Therefore, the amount of the magnetic charge "inside" the sphere must be quantized. It can be re-expressed as the requirement that $\int_{S^2} B = 2\pi N$, where N is an integer, because we had taken $e = \hbar = c = 1$. Second, we must use a complex coordinate to describe the sphere. The convenient choice is the so-called projective coordinate. You place a sphere on a plane, so that the south pole of the sphere touches the plane. Then draw a straight line from the north pole through the sphere down on the plane. One can use the coordinate of the complex plane to refer to the point on the sphere where the straight line goes through. The north pole is $z = \infty$. The volume two-form of the sphere is $\omega = idz \wedge d\bar{z}/(1 + z\bar{z})^2$, normalized as $\int_{S^2} \omega = 2\pi$. To switch to a coordinate that is regular at the north pole, one can attach another plane on the sphere, so that the sphere touches the plane at the north pole. The coordinate w on the second plane is related to that on the first plane as $w = 1/z$. It is easy to verify that the volume two-form ω is invariant under this coordinate change, $\frac{idz\wedge d\bar{z}}{(1+z\bar{z})^2} = \frac{idw\wedge d\bar{w}}{(1+w\bar{w})^2}$. The magnetic field is a two-form, $B = B_{z\bar{z}}dz \wedge d\bar{z} = N\omega$ so that its total flux through the sphere is $2\pi N$.

In the holormophic gauge, $B = N\omega = \bar{\partial}A$ with $\bar{A} = 0$, and hence $A = -Ni\bar{z}dz/(1+z\bar{z})$, regular at the pole $z = 0$. At the other pole $z \to \infty$, we can do the coordinate transformation $z = -1/w$, and $A = Nidw/w(1+w\bar{w})$, clearly singular at $w = 0$. We need to perform a gauge transformation $A' = A - iw^N \partial w^{-N} = -Ni\bar{w}dw/(1 + w\bar{w})$, a perfectly regular form. Therefore the required transition function is $w^N = (-1)^N z^{-N}$. The ground-state wave functions are holomorphic functions that transform under this transition function and are regular at both poles $z = 0$ and $w = 0$. The complete set of such functions are: $1, z, \cdots, z^N$, which transform to $w^N, w^{N-1}, \cdots, 1$ and hence are regular at both poles. There are $N + 1$ independent functions $\psi_k(z) = z^k$ for $k = 0, \cdots, N$. It is consistent with our original estimate $N \simeq \Phi/2\pi$.

To go back to the non-holomorphic but real gauge, we need

$$
\begin{aligned}
A' &= A - i(1+z\bar{z})^{N/2} d(1+z\bar{z})^{-N/2} \\
&= -N\frac{i\bar{z}dz}{1+z\bar{z}} + i\frac{N}{2}\frac{z\bar{z}+\bar{z}z}{1+z\bar{z}} \\
&= -\frac{N}{2}\frac{i(\bar{z}dz - zd\bar{z})}{1+z\bar{z}}.
\end{aligned}
\tag{A.8}
$$

This form is manifestly real, $A^* = A$. It requires the gauge change $\psi_k(z) \to \psi_k(z)(1+z\bar{z})^{-N/2}$. These are reminiscent of the wave functions on the plane in the symmetric gauge.

The inner product of the wave functions is therefore $\int \frac{\omega}{(1+z\bar{z})^N}\psi(z)^*\psi(z)$. It correctly accounts for the needed gauge transformation under the coordinate change $z = -1/w$,

$$
\begin{aligned}
\int \frac{\omega}{(1+z\bar{z})^N}\psi(z)^*\psi(z) &= \int \frac{\omega}{(1+(w\bar{w})^{-1})^N}\psi(-w^{-1})^*\psi(-w^{-1}) \\
&= \int \frac{\omega}{(1+w\bar{w})^N}(w^N\psi(-w^{-1}))^*(w^N\psi(-w^{-1})).
\end{aligned}
\tag{A.9}
$$

A.2 T^2

Another way of making the area finite is to impose a periodic boundary condition, by identifying $x \simeq x+1$, $y \simeq y+1$. This identification defines the two-dimensional torus T^2. We introduce the complex coordinate $z = x + \tau y$ ($\Im\tau > 0$). The parameter τ specifies the complex structure. The period is $z \to z+1$ and $z \to z+\tau$. The volume two-form is $\omega = 2\pi dx \wedge dy = 2\pi\frac{dz\wedge d\bar{z}}{\bar{\tau}-\tau}$. The magnetic field is $B = 2\pi N\frac{dz\wedge d\bar{z}}{\bar{\tau}-\tau}$, and $A = -2\pi N\frac{(\bar{z}-z)dz}{\bar{\tau}-\tau}$ so that it is invariant under $z \to z+1$. For the other period $z \to z+\tau$, $A \to A - 2\pi N dz$. It can be compensated by the gauge transformation $A \to A + \partial(2\pi N z + c)$, where c is a constant. Therefore, the wave function must also be gauge transformed by $e^{i2\pi N z + ic}$ for $z \to z+\tau$. Note that the gauge transformation has the correct periodicity for $z \to z+1$ despite its non-trivial z-dependence. We look for holomorphic functions that are periodic under $z \to z+1$ and quasi-periodic under $z \to z+\tau$ with the change by $e^{-i2\pi Nz - ic}$.

One parameter Θ-functions of level N are defined by

$$
\Theta_{m,N}(z) = \sum_{j\in\mathbb{Z}+\frac{m}{N}} e^{\pi i\tau N j^2} e^{2\pi i N j z}.
\tag{A.10}
$$

There are N independent Θ-functions for $m = 0, \cdots, N-1$. They obviously satisfy $\Theta_{m,N}(z+1) = \Theta_{m,N}(z)$ because $Nj \in \mathbb{Z}$ and hence $e^{2\pi i N j(z+1)} = e^{2\pi i N j z}$. The quasi-periodicity for $z \to z+\tau$ can be seen as

$$\begin{aligned}
\Theta_{m,N}(z+\tau) &= \sum_{j\in\mathbb{Z}+\frac{m}{N}} e^{\pi i \tau N j^2} e^{2\pi i N j(z+\tau)} \\
&= \sum_{j\in\mathbb{Z}+\frac{m}{N}} e^{\pi i \tau N j^2 + 2\pi i \tau N j} e^{2\pi i N j z} \\
&= \sum_{j\in\mathbb{Z}+\frac{m}{N}} e^{\pi i \tau N (j+1)^2 - \pi i \tau N} e^{2\pi i N j z} \\
&= e^{-\pi i \tau N} \sum_{j\in\mathbb{Z}+\frac{m}{N}} e^{\pi i \tau N j^2} e^{2\pi i N (j-1) z} \\
&= e^{-\pi i \tau N} e^{-2\pi i N z} \sum_{j\in\mathbb{Z}+\frac{m}{N}} e^{\pi i \tau N j^2} e^{2\pi i N j z} \\
&= e^{-\pi i \tau N} e^{-2\pi i N z} \Theta_{m,N}(z).
\end{aligned} \tag{A.11}$$

Therefore, it has exactly the correct gauge transformation properties under $z \to z+1$ (trivial) and $z \to z+\tau$.

To go back to the non-holomorphic but real gauge, we need

$$\begin{aligned}
A' &= A - i e^{i\pi N(\bar{z}-z)^2/2(\bar{\tau}-\tau)} d e^{-i\pi N(\bar{z}-z)^2/2(\bar{\tau}-\tau)} \\
&= -2\pi N \frac{(\bar{z}-z)dz}{\bar{\tau}-\tau} - \pi N \frac{(\bar{z}-z)(d\bar{z}-dz)}{\bar{\tau}-\tau} \\
&= -\pi N \frac{(\bar{z}-z)(d\bar{z}+dz)}{\bar{\tau}-\tau}.
\end{aligned} \tag{A.12}$$

This form is manifestly real, $A^* = A$. It requires the gauge change $\psi_k(z) \to \psi_k(z) e^{-i\pi N(\bar{z}-z)^2/2(\bar{\tau}-\tau)}$. These are reminiscent of the wave functions on the plane in the translationally invariant gauge. In fact, going back to the real coordinates $z = x + \tau y$, and specializing to the simple case $\tau = i$,

$$\begin{aligned}
&\Theta_{m,N}(z) e^{-i\pi N(\bar{z}-z)^2/2(\bar{\tau}-\tau)} \\
&\quad = \sum_{j\in\mathbb{Z}+\frac{m}{N}} e^{\pi i \tau N j^2} e^{2\pi i N j z} e^{-i\pi N(\bar{z}-z)^2/2(\bar{\tau}-\tau)} \\
&\quad = \sum_{j\in\mathbb{Z}+\frac{m}{N}} e^{-\pi N j^2} e^{2\pi i N j(x+iy)} e^{-\pi N y^2} \\
&\quad = \sum_{j\in\mathbb{Z}+\frac{m}{N}} e^{2\pi i N j x} e^{-\pi N (y+j)^2}.
\end{aligned} \tag{A.13}$$

Each term is precisely of the form Eq. (4.4) in the translationally invariant gauge (with x and y interchanged), summed up over j to force the periodicity under $x \to x+1$.

The inner product of wave functions is

$$\int \omega e^{-i\pi N(\bar{z}-z)^2/(\bar{\tau}-\tau)} \psi(z)^* \psi(z), \tag{A.14}$$

and indeed this definition is invariant under the change $z \to z+\tau$,

$$\begin{aligned} &\int \omega e^{-i\pi N(\bar{z}+\bar{\tau}-z-\tau)^2/(\bar{\tau}-\tau)} \psi(z+\tau)^* \psi(z+\tau) \\ &\quad = \int \omega e^{-i\pi N(\bar{z}-z)^2/(\bar{\tau}-\tau)} e^{-i2\pi N(\bar{z}-z)} e^{-i\pi N(\bar{\tau}-\tau)} \psi(z+\tau)^* \psi(z+\tau) \\ &\quad = \int \omega e^{-i\pi N(\bar{z}-z)^2/(\bar{\tau}-\tau)} (e^{i\pi N\tau} e^{i2\pi Nz} \psi(z+\tau))^* e^{i\pi N\tau} e^{i2\pi Nz} \psi(z+\tau) \\ &\quad = \int \omega e^{-i\pi N(\bar{z}-z)^2/(\bar{\tau}-\tau)} \psi(z)^* \psi(z). \end{aligned} \tag{A.15}$$

Here, we used the quasi-periodicity Eq. (A.11).

A.3 General genus

The mathematical question is: how many holomorphic sections are there for a twisted holomorphic line bundle on a Riemann surface? The answer is given by the index theorem for the twisted Dolbeaux complex and the Kodaira-Nakano vanishing theorem. The index theorem says that the number of holomorphic sections of $\Lambda^{(0,0)}$ type minus that of $\Lambda^{(1,0)}$ type is given by the Chern and Todd class $\int \mathrm{ch}(V) \wedge \mathrm{Td}(M) = c_1 - g$. Here c_1 is the first Chern class $c_1 = \frac{1}{2\pi} \int F = \frac{\Phi}{2\pi}$, and g is the genus of the Riemann surface. For instance, $g = -1$ for a sphere, $g = 0$ for a torus (*i.e.*, periodic boundary condition). Fortunately, there are no holormophic section of $\Lambda^{(1,0)}$ type according to the vanishing theorem and hence this index is nothing but the number of holormophic sections of $\Lambda^{(0,0)}$ type we are looking for. See, *e.g.*, T. Eguchi, P.B. Gilkey, and A.J. Hanson, "Gravitation, Gauge Theories, and Differential Geometry," *Phys. Rept.* **66**, 213 (1980), p. 118, and P. Griffiths and J. Harris, "Principles of Algebraic Geometry," Wiley (1978). Apart from the genus dependence, this precise result is consistent with our original estimate $N \simeq \Phi/2\pi$.