HW #4 (221A), due Sep 22, 4pm

1. The uncertainty relation is actually there in classical waves, too.

   (a) Show that a "localized light" of the form

   $$E_y(x,t) = E_0 \sin 2\pi\nu \left(t - \frac{x}{c}\right) e^{-(x-ct)^2/2\sigma^2} \tag{1}$$

   is a solution to the (one-dimensional) Maxwell equation. Sketch its shape. Calculate the "uncertainty" in the position Δx.

   (b) Using Fourier analysis, determine the frequency of the light and its dispersion $\Delta\nu$. What is the product $\Delta x \Delta\nu$, and how small can it be?

   Note However, the "uncertainty principle" in this case is purely classical, without involving $\hbar$. Only when you want to interpret the frequency ν as the momentum of the photon $p = \hbar\nu/c$, it becomes the quantum mechanical uncertainty principle.

2. The Gaussian wave packet represents a particle traveling in a "tight pack,"

   $$\psi(x) = \langle x|\psi\rangle = N e^{ipx/\hbar} e^{-(x-x_0)^2/4d^2}. \tag{2}$$

   Below, X and P are operators while x and p are numbers.

   (a) Work out the normalization constant N.

   (b) Show that $\langle X\rangle = x_0$.

   (c) Calculate ΔX.

   (d) Work out the wave function in the momentum space, $\phi(p) = \langle p|\psi\rangle$.

   (e) Show that $\langle P\rangle = p$.

   (f) Calculate ΔP and show this wave function is a "minimum uncertainty state," $\Delta X \Delta P = \hbar/2$.

3. The operator $U(a) = e^{ipa/\hbar}$ is a translation operator in space (here we consider only one dimension). To see this, we need to prove an identity

   $$\begin{aligned} e^A B e^{-A} &= \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{[A,[A,\cdots[A,}_{n} B \underbrace{]\cdots]]}_{n} \\ &= B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots \end{aligned} \tag{3}$$

(a) Consider $B(t) = e^{tA}Be^{-tA}$, where t is a real parameter. Show $\frac{d}{dt}B(t) = e^{tA}[A, B]e^{-tA}$.

(b) Obviously, $B(0) = B$ and therefore

$$pB(1) = B + \int_0^1 dt \frac{d}{dt} B(t). \tag{4}$$

Now using the power series $B(t) = \sum_{n=0}^{\infty} t^n B_n$ and using the above integral expression, show $B_n = \frac{1}{n}[A, B_{n-1}]$.

(c) Show by induction that

$$B_n = \frac{1}{n!} \underbrace{[A, [A, \cdots [A,}_{n} B \underbrace{] \cdots]]}_{n}.$$

(d) Use $B(1) = e^A B e^{-A}$ and prove the identity Eq. (3).

(e) Prove $e^{ipa/\hbar} x e^{-ipa/\hbar} = x + a$, showing $U(a)$ indeed translates space.