

# 221B Lecture Notes

## Spontaneous Symmetry Breaking

### 1 Spontaneous Symmetry Breaking

Spontaneous Symmetry Breaking is an ubiquitous concept in modern physics, especially in condensed matter and particle physics. It is a phenomenon where the fundamental Hamiltonian has a symmetry and hence has a conserved quantity due to Noether's theorem, but the ground state of the system does not respect the symmetry. There are many systems that exhibit this phenomenon: Bose–Einstein condensate, superfluid, superconductor, ferromagnet, anti-ferromagnet, any crystal, and according to the Standard Model of particle physics, the whole Universe.

To explain the idea, let me start with Heisenberg's model of ferromagnet. Then I go on to discuss field-theory examples. The purpose of this notes is not to discuss each example in detail, but rather try to identify a universal theme among them.

We will see that all of them exhibit similar behaviors:

1. Macroscopic coherence
2. Degenerate ground states

The Hamiltonian of Heisenberg's model of ferromagnets is very simple,

$$H = -J \sum_{\langle i,j \rangle} \vec{s}_i \cdot \vec{s}_j, \quad (1)$$

with  $J > 0$ , where the summation is over nearest neighbors on a cubic lattice. This is a phenomenological model of ferromagnet. For instance, if there is an unpaired electron for each atom at the lattice sites, Coulomb repulsion between electrons would prefer anti-symmetric spatial wave function, and hence symmetric spin wave function. Therefore, unpaired spins on neighboring sites want to be aligned. Note that the Hamiltonian is perfectly rotationally symmetric. If you rotate all spins by the same amount, Hamiltonian does not change at all. According to Noether's theorem, a symmetry of the system must lead to a conserved quantity. In this case, it is the total spin of the system,

$$\vec{S} = \sum_i \vec{s}_i, \quad (2)$$

that commutes with the Hamiltonian  $[\vec{S}, H] = 0$  and hence is conserved. The rotation of all spins is then generated by the spin operator, namely

$$U(\vec{\theta}) = e^{i\vec{S}\cdot\vec{\theta}/\hbar} \quad (3)$$

rotates all spins around the vector  $\vec{\theta}$  by the angle  $|\vec{\theta}|$ .

It is useful to rewrite the Hamiltonian as

$$H = -J \sum_{\langle i,j \rangle} \left( s_i^z s_j^z + \frac{1}{2} (s_i^+ s_j^- + s_i^- s_j^+) \right). \quad (4)$$

Using this form, it is easy to see that the state with all spins up

$$|\cdots \uparrow \uparrow \cdots \rangle \quad (5)$$

is an eigenstate of the Hamiltonian. This is what is called spontaneous breaking of symmetry. The Hamiltonian is perfectly symmetric under rotation of spins. However, the ground state “spontaneously” chooses a particular orientation and hence is not invariant under the symmetry (rotation).

To simplify the discussion, let us imagine a spin chain, *i.e.*, a one-dimensional lattice of  $N$  sites with periodic boundary condition. We take the limit  $N \rightarrow \infty$  at the end of calculations. (There is an important subtlety about one-dimensional quantum systems which actually do not show spontaneous symmetry breaking due to long-range quantum fluctuations. This is known as Mermin–Wagner theorem in condensed matter physics or Coleman’s theorem in 1+1 dimensional quantum field theory. I cannot go into this issue in this lecture notes, but I have to warn you that what we discuss here uses one-dimensional spin chain only for the sake of simplicity of calculations, and we actually imagine 3-D system in our mind.) The energy eigenvalue is

$$H|\uparrow \cdots \uparrow\rangle = -JN \left(\frac{\hbar}{2}\right)^2 \quad (6)$$

because there are  $N$  nearest neighbor pairs among  $N$  spins in a chain. This is indeed the ground state of the system. Using the standard addition of spins, this state has total spin  $N\hbar/2$ , and hence comes with a multiplicity of  $N+1$ . They correspond to different orientation of the spin  $N/2$ . As  $N \rightarrow \infty$ , there are many many possible orientations close to each other, and eventually it becomes continuous as a classical angular momentum. Mathematically, if you

define the orientation of the spin vector by  $\vec{n} = \vec{S}/(N/2)$ , their commutation relations are

$$[n^i, n^j] = \frac{1}{(N/2)^2} [S^i, S^j] = \frac{1}{(N/2)^2} i\hbar \epsilon^{ijk} S^k = \frac{i\hbar}{N/2} \epsilon^{ijk} n^k. \quad (7)$$

As  $N \rightarrow 0$ , it is the same as  $\hbar \rightarrow 0$  limit, and hence the ‘‘orientation vector’’ becomes classical. Therefore, the orientation of the maroscopic spin of the ground state can be regarded as a classical object.

One criterion for the classical spin vector is that there must be infinitely many possible orientations of spin. It is easy to see that there are infinitely many possible orientations. Starting from the state where all spins are up, we perform finite rotations of all spins using the unitarity operator Eq. (3). Recall the formula back in 221A,

$$e^{i\vec{\sigma}\cdot\vec{\theta}/\hbar} = e^{i\vec{\sigma}\cdot\vec{\theta}/2} = \cos \frac{|\vec{\theta}|}{2} + i\vec{\sigma} \cdot \frac{\vec{\theta}}{|\vec{\theta}|} \sin \frac{|\vec{\theta}|}{2}. \quad (8)$$

We find

$$e^{i\vec{\sigma}\cdot\vec{\theta}/2} |\uparrow\rangle = \cos \frac{|\vec{\theta}|}{2} |\uparrow\rangle + i \sin \frac{|\vec{\theta}|}{2} \left( \frac{\theta_z}{|\vec{\theta}|} |\uparrow\rangle + \frac{\theta_x + i\theta_y}{|\vec{\theta}|} |\downarrow\rangle \right). \quad (9)$$

Therefore,

$$\langle \cdots \uparrow \uparrow \cdots | U(\theta) | \cdots \uparrow \uparrow \cdots \rangle = \lim_{N \rightarrow \infty} \left( \cos \frac{|\vec{\theta}|}{2} + i \frac{\theta_z}{|\vec{\theta}|} \sin \frac{|\vec{\theta}|}{2} \right)^N. \quad (10)$$

Unless  $\vec{\theta} \parallel z$ , the terms in the parentheses have modulus less than unity,

$$\left| \cos \frac{|\vec{\theta}|}{2} + i \frac{\theta_z}{|\vec{\theta}|} \sin \frac{|\vec{\theta}|}{2} \right|^2 = 1 - \frac{\theta_x^2 + \theta_y^2}{|\vec{\theta}|^2} \sin^2 \frac{|\vec{\theta}|}{2} < 1 \quad (11)$$

and the limit is zero. In other words, a rotation of a ground state yields a different ground state  $U(\theta) |\cdots \uparrow \uparrow \cdots\rangle$  that is orthogonal to the original ground state. There are infinitely many ground states related to each other by rotation.

In general, suppose the Hamiltonian is invariant under a unitarity transformation  $U$ :

$$UHU^{-1} = H, \quad (12)$$

but the ground state isn't,

$$U|0\rangle \neq |0\rangle. \tag{13}$$

Then the system is said to have spontaneous symmetry breaking. Quantum system exhibits macroscopic coherence in this case.

There are many examples of spontaneous symmetry breaking. The potential with positive chemical potential we discussed for Bose–Einstein condensate, the “wine-bottle” or “Mexican hat” potential is a typical example of the potential that leads to spontaneous symmetry breaking. The ferromagnet discussed above is another example. Yet another example is crystals. When liquid cools and becomes solid, the atoms line up in a crystalline structure. Liquid, of course, is a translationally invariant system. The interatomic attraction due to van der Waals force makes all atoms behave as a continuous medium, but the overall translation of the system does not make any difference. However, in a crystal, atoms are situated at particular locations and an overall translation takes it to another different crystal. Translational invariance is spontaneously broken. Also is rotational invariance broken because a, say, cubic lattice orients itself along a particular lattice vector.

When the symmetry that is spontaneously broken is continuous, such as rotation of spins, translation of the lattice, or phase change in the Schrödinger field, there are infinitely many ground states. If the symmetry is discrete, such as spin inversion in Ising model, there are only finite number of ground states.

In the case of spontaneously broken continuous symmetry, there is always a low-lying excitation whose energy  $E(\vec{p})$  does not have a gap  $E(\vec{p}) \rightarrow 0$  as its momentum goes to zero  $\vec{p} \rightarrow 0$ . In the case of liquid superfluid  $^4\text{He}$ , phonon energy indeed goes to zero linearly. Similarly, phonon, the quantized sound wave of a lattice, and phonon excitations we discussed above Bose–Einstein condensate, also go linearly to zero. This is a completely universal phenomenon with a simple reason. Imagine you attach the “wine-bottle” potential at every point in space. The minimum of the potential spontaneously chooses one point along the bottom of the bottle. If the choice is the same everywhere in space, this is a ground state configuration. If the choice of the minimum is a little bit different from a point to point, there is spatial derivatives, which is nothing but the kinetic energy in the system because  $\vec{p} = -i\hbar\vec{\nabla}$ . On the other hand, there is no cost in the potential energy. By making the spatial variation less and less, it eventually goes to one of the ground states, and hence there should not be a gap in the excitation energy.

We can verify this general argument in the case of ferromagnet. Consider a state

$$|k\rangle = \sum_n |\cdots \uparrow \cdots \downarrow^n \cdots \uparrow \cdots\rangle e^{ikna}. \quad (14)$$

Here,  $a$  is the lattice constant, and  $k$  is the wave vector. Under an overall translation by a lattice constant,  $e^{ipa/\hbar}$ , the state responds as

$$\begin{aligned} e^{ipa/\hbar}|k\rangle &= \sum_n |\cdots \uparrow \cdots \downarrow^{n-1} \cdots \uparrow \cdots\rangle e^{ikna} \\ &= \sum_n |\cdots \uparrow \cdots \downarrow^n \cdots \uparrow \cdots\rangle e^{ik(n+1)a} \\ &= e^{ika}|k\rangle. \end{aligned} \quad (15)$$

Therefore, this state has a finite momentum  $p = \hbar k$  and hence  $k$  is a wave vector. The energy of the state is read off from

$$\begin{aligned} H|k\rangle &= -J \sum_m \left( s_m^z s_{m+1}^z + \frac{1}{2} (s_m^+ s_{m+1}^- + s_m^- s_{m+1}^+) \right) \sum_n |\cdots \uparrow \cdots \downarrow^n \cdots \uparrow \cdots\rangle e^{ikna}. \end{aligned} \quad (16)$$

First, the  $s_i^z s_j^z$  terms give

$$\begin{aligned} &-J \sum_m s_m^z s_{m+1}^z \sum_n |\cdots \uparrow \cdots \downarrow^n \cdots \uparrow \cdots\rangle e^{ikna} \\ &= -J \left( (N-2) \left( \frac{\hbar}{2} \right)^2 + 2 \left( \frac{\hbar}{2} \right) \left( -\frac{\hbar}{2} \right) \right) |k\rangle. \end{aligned} \quad (17)$$

On the other hand, the raising and lowering operators give

$$\begin{aligned} &-J \sum_m \frac{1}{2} (s_m^+ s_{m+1}^- + s_m^- s_{m+1}^+) \sum_n |\cdots \uparrow \cdots \downarrow^n \cdots \uparrow \cdots\rangle e^{ikna} \\ &= -J \sum_m \frac{\hbar^2}{2} \left( |\cdots \uparrow \cdots \downarrow^{m+1} \cdots \uparrow \cdots\rangle e^{ikma} + |\cdots \uparrow \cdots \downarrow^{m-1} \cdots \uparrow \cdots\rangle e^{ikma} \right) \\ &= -J \frac{\hbar^2}{2} (e^{-ika} + e^{ika}) |k\rangle. \end{aligned} \quad (18)$$

Putting both terms together, we find

$$H|k\rangle = -J \left( N \frac{\hbar^2}{4} + \hbar^2 (\cos ka - 1) \right) |k\rangle, \quad (19)$$

and hence the excitation energy is

$$E(k) = J(1 - \cos ka). \quad (20)$$

Consistent with the general discussion, the excitation energy smoothly goes to zero without a gap as the momentum goes to zero, even though the dispersion relation is quadratic in momentum rather than linear.