221B HW #5, due Feb 18 (Fri), 4pm

1. Study the scattering by a not-so-hard sphere $V(r) = \frac{2m}{\hbar^2} K^2$ $(r < a)$ and $V(r) = 0$ $(r > a)$. Assume $k > K$.

   (a) Solve for the phase shifts exactly. Plot $\sin^2 \delta_l$ for $Ka = 10$ and 3 and $ka = 30$ up to $l = 50$. Calculate the total cross sections.

   (b) Use the semi-classical formula to do the same and compare them.

   (c) Show that the expansion of the semi-classical formula to $O(K^2)$ is nothing but the eikonal approximation Eq. (7.4.14) (p. 394) in Sakurai (see also p. 404).

   (d) Use the Born approximation to calculate the total cross section and compare them.

2. The Gaussian wave packet outside the scattering region is given by

$$rR_l(r,t) = \frac{d}{\sqrt{2\pi}} \int dq\, e^{-(q-k)^2 d^2/2} \left(e^{iqr} e^{2i\delta_l(q)} - (-1)^l e^{-iqr} \right) e^{-i\hbar q^2 t/2m}. \qquad (1)$$

   Close to a resonance, the phase shift is well approximated by $e^{2i\delta_l(q)} = \frac{q-k_0-i\kappa}{q-k_0+i\kappa}$. Follow the steps to study the behavior of the wave packet.

   (a) First study the case with no scattering $e^{2i\delta_l(q)} = 1$. Expand the phase around $q = k$ up to the first order in $(q-k)$, perform the Gaussian integral and show that

$$rR_l(r,t) = e^{-i\hbar k^2 t/2m} (e^{ikr} e^{-(r-vt)^2/2d^2} - (-1)^l e^{-ikr} e^{-(r+vt)^2/2d^2}), \qquad (2)$$

   where $v = \hbar k/m$. Note that the outgoing (incoming) wave is appreciable only for $vt = r > 0$ $(vt = -r < 0)$.

   (b) Now study the remaining piece proportional to $-2i\kappa/(q - k_0 + i\kappa)$. Assume that the Gaussian is wider than the resonance, and factor out the Gaussian as $e^{-(k-k_0)^2 d^2/2}$ outside the integral. Expand the phase around $q = k_0$ up to the first order in $(q - k_0)$, and perform the integral. Show that the pole contributes only when $r < vt$.

   (c) Using results from (a) and (b), make a few plots to show that the wave is purely Gaussian for $t < 0$, while it has the "prompt" Gaussian piece and the "delayed" contribution from the resonance for $t > 0$.

   (d) Show that the lifetime of the delayed piece $\propto e^{-t/2\tau}$ is related to the imaginary part Γ of the energy $E = \hbar^2 k^2/2m = E_0 - i\Gamma/2$ at the pole as $\tau = \hbar/\Gamma$.

Note The semi-classical approximation is valid for large l. The phase shift is given by the difference in the classical action with and without the potential

$$\delta_l = \lim_{R\to\infty} \left[\int_{r_0}^{R} \sqrt{k^2 - U(r) - \frac{l^2}{r^2}}\, dr - \int_{l/k}^{R} \sqrt{k^2 - \frac{l^2}{r^2}}\, dr \right]. \tag{3}$$

Here, $U(r) = \frac{2m}{\hbar^2} V(r)$, and r_0 is the classical turning point where the argument of the square root in the integrand vanishes. The integral for the free case can be done exactly:

$$\int_{l/k}^{R} \sqrt{k^2 - \frac{l^2}{r^2}}\, dr = \sqrt{(kR)^2 - l^2} - 2l \arctan \frac{\sqrt{(kR)^2 - l^2}}{kR + l} = kR - \frac{l\pi}{2} + O(R^{-1}). \tag{4}$$

Note that this behavior is exactly what is seen in the asymptotic behavior of the spherical Bessel function $\rho j_l(\rho) \simeq \sin\left(\rho - \frac{l\pi}{2}\right)$.

In the literature, you also see expressions where l^2 is replaced by $(l + 1/2)^2$ or $l(l+1)$. They are formally indistinguishable in the large l limit.