HW #5

1. Not-so-hard sphere

(a)

We solve the problem exactly. The wave function is $R_l(r) = j_l(k\,r)\cos\delta_l + n_l(k\,r)\sin\delta_l$ for $r > a$, and $R_l(r) = j_l\left(\sqrt{k^2 - K^2}\ r\right)$ for $r < a$ at high energies $k > K = \sqrt{2\,m\,V}\,\big/\,\hbar$, . Requiring the logarithmic derivatives to match,

$$\text{Simplify}\Big[1\Big/\left(\sqrt{\frac{\pi}{2\,k\,r}}\ \text{BesselJ}\left[\frac{(2\,l+1)}{2},\,k\,r\right]\text{Cos}[\delta_l] + \sqrt{\frac{\pi}{2\,k\,r}}\ \text{BesselY}\left[\frac{(2\,l+1)}{2},\,k\,r\right]\text{Sin}[\delta_l]\right)$$
$$\text{D}\left[\sqrt{\frac{\pi}{2\,k\,r}}\ \text{BesselJ}\left[\frac{(2\,l+1)}{2},\,k\,r\right]\text{Cos}[\delta_l] + \sqrt{\frac{\pi}{2\,k\,r}}\ \text{BesselY}\left[\frac{(2\,l+1)}{2},\,k\,r\right]\text{Sin}[\delta_l],\,r\right]\Big]$$

$$-\Big(-k\,r\,\text{BesselJ}\left[-\tfrac{1}{2}+l,\,k\,r\right]\text{Cos}[\delta_l] + \text{BesselJ}\left[\tfrac{1}{2}+l,\,k\,r\right]\text{Cos}[\delta_l] + k\,r\,\text{BesselJ}\left[\tfrac{3}{2}+l,\,k\,r\right]\text{Cos}[\delta_l] - k\,r\,\text{BesselY}\left[-\tfrac{1}{2}+l,\,k\,r\right]\text{Sin}[\delta_l] + \text{BesselY}\left[\tfrac{1}{2}+l,\,k\,r\right]\text{Sin}[\delta_l] + k\,r\,\text{BesselY}\left[\tfrac{3}{2}+l,\,k\,r\right]\text{Sin}[\delta_l]\Big)\Big/\Big(2\,r\,\text{BesselJ}\left[\tfrac{1}{2}+l,\,k\,r\right]\text{Cos}[\delta_l] + 2\,r\,\text{BesselY}\left[\tfrac{1}{2}+l,\,k\,r\right]\text{Sin}[\delta_l]\Big)$$

$$\text{Simplify}\left[\frac{\text{D}\left[\sqrt{\frac{\pi}{2\sqrt{k^2-K^2}\ r}}\ \text{BesselJ}\left[\frac{(2\,l+1)}{2},\,\sqrt{k^2-K^2}\ r\right],\,r\right]}{\sqrt{\frac{\pi}{2\sqrt{k^2-K^2}\ r}}\ \text{BesselJ}\left[\frac{(2\,l+1)}{2},\,\sqrt{k^2-K^2}\ r\right]}\right]$$

$$-\Big(-\sqrt{k^2-K^2}\ r\,\text{BesselJ}\left[-\tfrac{1}{2}+l,\,\sqrt{k^2-K^2}\ r\right] + \text{BesselJ}\left[\tfrac{1}{2}+l,\,\sqrt{k^2-K^2}\ r\right] + \sqrt{k^2-K^2}\ r\,\text{BesselJ}\left[\tfrac{3}{2}+l,\,\sqrt{k^2-K^2}\ r\right]\Big)\Big/\Big(2\,r\,\text{BesselJ}\left[\tfrac{1}{2}+l,\,\sqrt{k^2-K^2}\ r\right]\Big)$$

```
sol =
 Solve[-(-k r BesselJ[-1/2 + l, k r] cot + BesselJ[1/2 + l, k r] cot + k r BesselJ[3/2 + l, k r] cot -
        k r BesselY[-1/2 + l, k r] + BesselY[1/2 + l, k r] + k r BesselY[3/2 + l, k r]) /
     (2 r BesselJ[1/2 + l, k r] cot + 2 r BesselY[1/2 + l, k r]) ==
    -(-Sqrt[k^2 - K^2] r BesselJ[-1/2 + l, Sqrt[k^2 - K^2] r] + BesselJ[1/2 + l, Sqrt[k^2 - K^2] r] + Sqrt[k^2 - K^2] r
        BesselJ[3/2 + l, Sqrt[k^2 - K^2] r]) / (2 r BesselJ[1/2 + l, Sqrt[k^2 - K^2] r]) /. {r -> a}, cot]
```

```
{{cot ->
   (-k BesselJ[1/2 + l, a Sqrt[k^2 - K^2]] BesselY[-1/2 + l, a k] + Sqrt[k^2 - K^2] BesselJ[-1/2 + l, a Sqrt[k^2 - K^2]]
       BesselY[1/2 + l, a k] - Sqrt[k^2 - K^2] BesselJ[3/2 + l, a Sqrt[k^2 - K^2]] BesselY[1/2 + l, a k] +
      k BesselJ[1/2 + l, a Sqrt[k^2 - K^2]] BesselY[3/2 + l, a k]) /
   (-Sqrt[k^2 - K^2] BesselJ[-1/2 + l, a Sqrt[k^2 - K^2]] BesselJ[1/2 + l, a k] + k BesselJ[-1/2 + l, a k]
       BesselJ[1/2 + l, a Sqrt[k^2 - K^2]] - k BesselJ[1/2 + l, a Sqrt[k^2 - K^2]] BesselJ[3/2 + l, a k] +
      Sqrt[k^2 - K^2] BesselJ[1/2 + l, a k] BesselJ[3/2 + l, a Sqrt[k^2 - K^2]])}}
```

```
sin2delta1 = Simplify[1/(1 + cot^2) /. sol[[1]]]
```

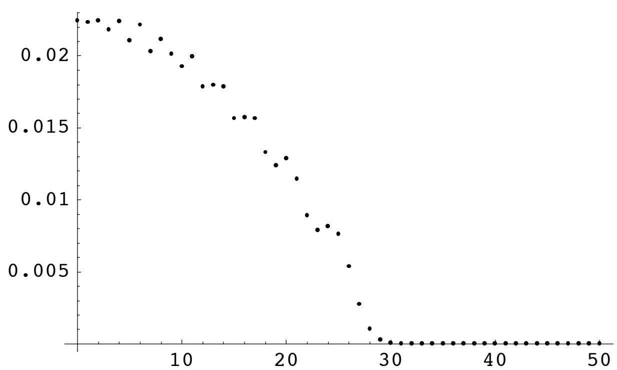
```
1 /
 (1 + (Sqrt[k^2 - K^2] (BesselJ[-1/2 + l, a Sqrt[k^2 - K^2]] - BesselJ[3/2 + l, a Sqrt[k^2 - K^2]]) BesselY[1/2 + l, a k] +
        k BesselJ[1/2 + l, a Sqrt[k^2 - K^2]] (-BesselY[-1/2 + l, a k] + BesselY[3/2 + l, a k]))^2 /
   (Sqrt[k^2 - K^2] BesselJ[-1/2 + l, a Sqrt[k^2 - K^2]] BesselJ[1/2 + l, a k] - k BesselJ[-1/2 + l, a k]
        BesselJ[1/2 + l, a Sqrt[k^2 - K^2]] + k BesselJ[1/2 + l, a Sqrt[k^2 - K^2]] BesselJ[3/2 + l, a k] -
       Sqrt[k^2 - K^2] BesselJ[1/2 + l, a k] BesselJ[3/2 + l, a Sqrt[k^2 - K^2]])^2)
```

For $K a = 3$,

```
table1 = Table[{l, sin2deltal /. {a → 1, k → 30., K → 3.}}, {l, 0, 50}]
```

{{0, 0.0224499}, {1, 0.0223369}, {2, 0.0224495}, {3, 0.0218459}, {4, 0.0224205}, {5, 0.0210577}, {6, 0.0221591}, {7, 0.0203273}, {8, 0.021162}, {9, 0.0201285}, {10, 0.0192687}, {11, 0.0199514}, {12, 0.0178872}, {13, 0.0179801}, {14, 0.0178838}, {15, 0.0156655}, {16, 0.0157336}, {17, 0.015663}, {18, 0.0133059}, {19, 0.0123831}, {20, 0.0128676}, {21, 0.0114542}, {22, 0.00892118}, {23, 0.00789587}, {24, 0.00817337}, {25, 0.00763292}, {26, 0.00538177}, {27, 0.00275659}, {28, 0.00103758}, {29, 0.000292554}, {30, 0.0000629833}, {31, 0.0000105541}, {32, 1.40307×10^{-6}}, {33, 1.50655×10^{-7}}, {34, 1.32748×10^{-8}}, {35, 9.73072×10^{-10}}, {36, 6.00373×10^{-11}}, {37, 3.15296×10^{-12}}, {38, 1.42497×10^{-13}}, {39, 5.47066×10^{-15}}, {40, 2.73194×10^{-16}}, {41, 3.0721×10^{-19}}, {42, 3.56932×10^{-17}}, {43, 4.20285×10^{-17}}, {44, 1.20598×10^{-18}}, {45, 5.95957×10^{-18}}, {46, 2.7534×10^{-17}}, {47, 1.06462×10^{-17}}, {48, 5.92185×10^{-18}}, {49, 2.98054×10^{-17}}, {50, 6.17154×10^{-18}}}

```
plot1 = ListPlot[table1]
```



- Graphics -

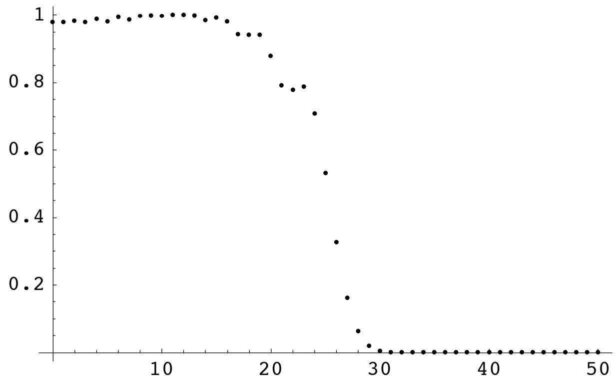
$$\texttt{sigma1 = Sum}\left[\frac{4\pi(2l+1)}{k^2}\ \texttt{table1[[l + 1, 2]] /. \{k → 30.\}, \{l, 0, 50\}}\right]$$

0.140557

```
table2 = Table[{l, sin2deltal /. {a → 1, k → 30., K → 10.}}, {l, 0, 50}]
```

{{0, 0.979313}, {1, 0.978979}, {2, 0.982717}, {3, 0.97905}, {4, 0.989201}, {5, 0.980278}, {6, 0.994967}, {7, 0.986532}, {8, 0.997096}, {9, 0.997642}, {10, 0.997112}, {11, 0.99917}, {12, 0.999995}, {13, 0.998418}, {14, 0.985572}, {15, 0.991908}, {16, 0.981422}, {17, 0.942118}, {18, 0.941285}, {19, 0.9421}, {20, 0.877756}, {21, 0.7907}, {22, 0.777905}, {23, 0.787805}, {24, 0.707436}, {25, 0.532233}, {26, 0.326976}, {27, 0.161754}, {28, 0.0636946}, {29, 0.0196975}, {30, 0.00472489}, {31, 0.000875119}, {32, 0.000126084}, {33, 0.0000143711}, {34, 1.32218×10^{-6}}, {35, 1.00029×10^{-7}}, {36, 6.32096×10^{-9}}, {37, 3.38766×10^{-10}}, {38, 1.54618×10^{-11}}, {39, 5.04936×10^{-13}}, {40, 2.47407×10^{-14}}, {41, 1.66814×10^{-14}}, {42, 7.72163×10^{-15}}, {43, 2.95593×10^{-16}}, {44, 7.74072×10^{-18}}, {45, 4.63864×10^{-17}}, {46, 3.53274×10^{-17}}, {47, 2.8348×10^{-16}}, {48, 2.54894×10^{-17}}, {49, 2.95414×10^{-17}}, {50, 1.095×10^{-17}}}

```
plot2 = ListPlot[table2]
```




```
- Graphics -
```

$$\texttt{sigma2 = Sum}\left[\frac{4\pi(2l+1)}{k^2}\ \texttt{table2[[l + 1, 2]] /. \{k → 30.\}, \{l, 0, 50\}}\right]$$

```
8.74545
```

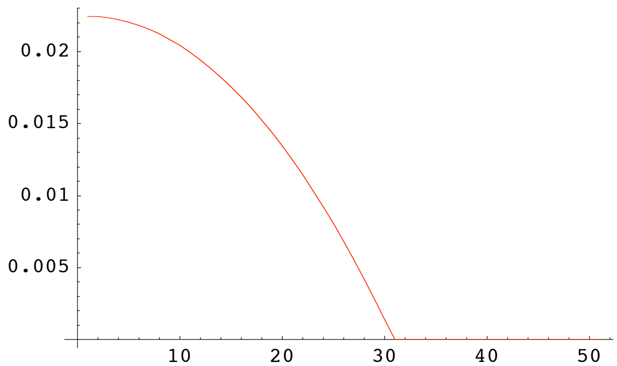
(b)

In the semi-classical formula, the term with the potential is obtained from that without the potential by the replacement $k \to \sqrt{k^2 - K^2}$. Therefore, $\delta_l = \left(\sqrt{(k^2-K^2)\,a^2 - l^2} - 2\,l \arctan \frac{\sqrt{(k^2-K^2)\,a^2-l^2}}{\sqrt{k^2-K^2}\ a+l}\right) - \left(\sqrt{k^2\,a^2 - l^2} - 2\,l \arctan \frac{\sqrt{k^2\,a^2-l^2}}{k\,a+l}\right)$
Note that the term should be dropped when the argument of the square root is negative because it implies there is no integration region.

$$\texttt{table3 = Table}\left[\texttt{Sin}\left[\left(\sqrt{\texttt{Max}[a^2\ (k^2 - K^2) - l^2,\ 0]} - 2\,l\,\texttt{ArcTan}\left[\frac{\sqrt{\texttt{Max}[a^2\ (k^2-K^2) - l^2,\ 0]}}{a\sqrt{k^2-K^2} + l}\right] - \sqrt{\texttt{Max}[k^2\,a^2 - l^2,\ 0]} + 2\,l\,\texttt{ArcTan}\left[\frac{\sqrt{\texttt{Max}[k^2\,a^2 - l^2,\ 0]}}{a\,k + l}\right]\right)\right]^2 \texttt{ /. \{a → 1, k → 30., K → 3.\}, \{l, 0, 50\}}\right]$$

```
{0.0224433, 0.0224184, 0.0223438, 0.0222194, 0.0220452, 0.0218213, 0.0215476,
 0.021224, 0.0208505, 0.0204271, 0.0199538, 0.0194305, 0.0188571, 0.0182336,
 0.01756, 0.0168361, 0.0160619, 0.0152372, 0.0143621, 0.0134365, 0.0124601,
 0.0114331, 0.0103551, 0.00922615, 0.00804612, 0.00681487, 0.00553227, 0.00419818,
 0.00281242, 0.00137465, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

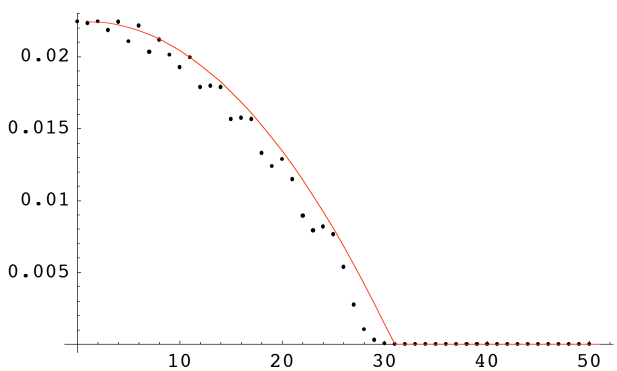
plot3 = ListPlot[table3, PlotJoined → True, PlotStyle → {RGBColor[1, 0, 0]}]



- Graphics -

Compared to the exact result, it is quite close.

Show[plot1, plot3]



- Graphics -

For the total cross section, compared to the exact result 0.140557,it is agian quite close, with about 5% error.

$$\text{sigma3} = \text{Sum}\left[\frac{4\,\pi\,(2\,l+1)}{k^2}\ \text{table3}[[l+1]]\ /.\ \{k \to 30.\},\ \{l,\ 0,\ 50\}\right]$$

0.146963

$$\frac{\text{sigma3} - \text{sigma1}}{\text{sigma1}}$$

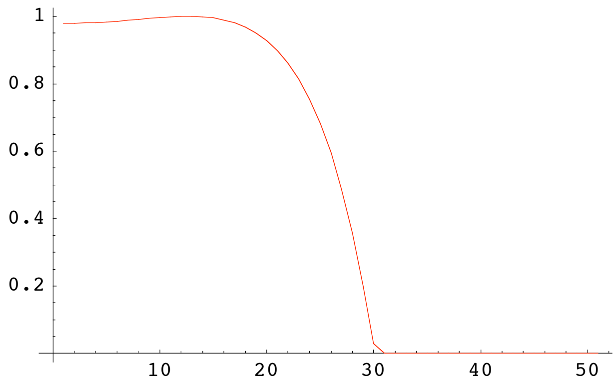
0.045582

Now for $K\,a = 10$,

```
table4 = Table[Sin[
    (√Max[a² (k² - K²) - l², 0] - 2 l ArcTan[√Max[a² (k² - K²) - l², 0] / (a √(k² - K²) + l)] - √Max[k² a² - l², 0] +
     2 l ArcTan[√Max[k² a² - l², 0] / (a k + l)])]² /. {a → 1, k → 30., K → 10.}, {l, 0, 50}]
```

```
{0.979141, 0.979429, 0.980283, 0.981669, 0.983533, 0.985795, 0.988351, 0.991067, 0.993779,
 0.996285, 0.998348, 0.99968, 0.999945, 0.998746, 0.995619, 0.990022, 0.981322, 0.968786,
 0.95156, 0.928656, 0.898932, 0.861062, 0.813517, 0.754526, 0.682035, 0.593648, 0.486513,
 0.357, 0.198382, 0.0294356, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

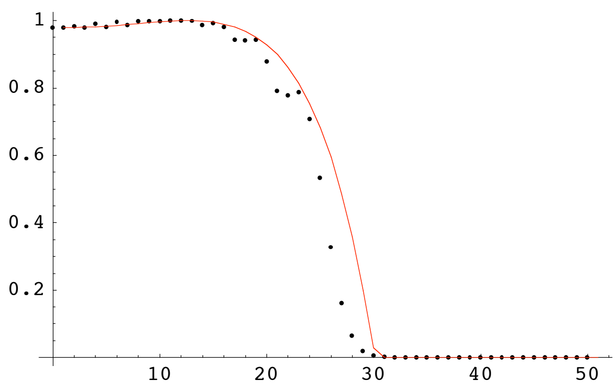
```
plot4 = ListPlot[table4, PlotJoined → True, PlotStyle → {RGBColor[1, 0, 0]}]
```



- Graphics -

Compared to the exact result, it is quite close

```
Show[plot2, plot4]
```



- Graphics -

For the total cross section, compared to the exact result 8.74545,it is agian quite close, with about 5% error.

```
sigma4 = Sum[(4 π (2 l + 1))/k² table4[[l + 1]] /. {k → 30.}, {l, 0, 50}]
```

```
9.2187
```

```
(sigma4 - sigma2)/sigma2
```

0.0541137

(c)

The eikonal approximation in Sakurai relates the phase shift to $\delta_l = \Delta(b)\,|_{b=l/k}$ (7.6.24) where $\Delta(b) = -\frac{m}{2 k \hbar^2} \int_{-\infty}^{\infty} V\left(\sqrt{b^2 + z^2}\right) d z$. In our case, we only need to know the distance the straight line with impact parameter b in Figure 7.5 (page 393) goes through the radius r, which is $2\sqrt{a^2 - b^2}$. Therefore, $\Delta(b) = -\frac{m}{2 k \hbar^2} V\, 2\sqrt{a^2 - b^2} = -\frac{K^2}{2k} \sqrt{a^2 - b^2}$. The result is hence $\delta_l = -\frac{K^2}{2k} \sqrt{a^2 - (l/k)^2} = -\frac{K^2}{2k^2} \sqrt{k^2 a^2 - l^2}$.

Expand the semi-classical formula,

```
Simplify[Series[(Sqrt[(k^2 - K^2) a^2 - l^2] - 2 l ArcTan[Sqrt[(k^2 - K^2) a^2 - l^2]/(Sqrt[k^2 - K^2] a + l)]) -
    (Sqrt[k^2 a^2 - l^2] - 2 l ArcTan[Sqrt[k^2 a^2 - l^2]/(k a + l)]), {K, 0, 2}]]
```

$$2\, l \left(\text{ArcTan}\left[\frac{\sqrt{a^2 k^2 - l^2}}{a k + l}\right] - \text{ArcTan}\left[\frac{\sqrt{a^2 k^2 - l^2}}{a \sqrt{k^2} + l}\right]\right) - \frac{\sqrt{a^2 k^2 - l^2}\, K^2}{2 k^2} + O[K]^3$$

```
PowerExpand[%]
```

$$-\frac{\sqrt{a^2 k^2 - l^2}\, K^2}{2 k^2} + O[K]^3$$

This is precisely the result from the eikonal approximation.

In general, the eikonal approximation is a simplified formula of the semi-classical result when the potential is weak compared to the kinetic energy.

(d)

The Born approximation says

$$f^{(1)} = -\frac{2m}{\hbar^2} \frac{1}{q} \int_0^\infty r\, V(r) \sin q r\, d r$$

$$= -\frac{2m}{\hbar^2} \frac{1}{q} \int_0^a r \frac{\hbar^2 K^2}{2m} \sin q r\, d r = -\frac{K^2}{q} \frac{\sin q a - q a \cos q a}{q^2}$$

```
Integrate[r Sin[q r], {r, 0, a}]
```

$$\frac{-a\, q \,\text{Cos}[a\, q] + \text{Sin}[a\, q]}{q^2}$$

Note that $q^2 = 2 k^2 (1 - \cos\theta)$ and hence $d\cos\theta = \frac{1}{2k^2}\, d q^2 = \frac{q}{k^2}\, d q$. The cross section is then

```
sigmaBorn = Simplify[Integrate[2 π (K^2 (-a q Cos[a q] + Sin[a q])/q^3)^2 q/k^2, {q, 0, 2 k}]]
```

$$\frac{K^4 \pi \left(-1 - 8 a^2 k^2 + 32 a^4 k^4 + \text{Cos}[4 a k] + 4 a k \,\text{Sin}[4 a k]\right)}{64 k^6}$$

```
sigmaBorn /. {k → 30, a → 1, K → 3.}
```

```
0.141333
```

$$\frac{\% - \text{sigma1}}{\text{sigma1}}$$

```
0.00552286
```

This is very close, as good as 0.6%.

```
sigmaBorn /. {k → 30, a → 1, K → 10.}
```

```
17.4485
```

$$\frac{\% - \text{sigma2}}{\text{sigma2}}$$

```
0.995151
```

This one has 100% error! As K is increased, it goes beyond the validity of the Born approximation. In comparison, the semi-classical formula remained very good.

2. Gaussian Wave Packet with Resonance

(a)

$$\frac{d}{\sqrt{2\pi}} \int_0^\infty e^{-(q-k)^2 d^2/2} (e^{i q r} e^{2 i \delta} - (-1)^l e^{-i q r}) e^{-i \hbar q^2 t/2 m} \, d q$$

Without the scattering, it is

$$\frac{d}{\sqrt{2\pi}} \int_0^\infty e^{-(q-k)^2 d^2/2} (e^{i q r} - (-1)^l e^{-i q r}) e^{-i \hbar q^2 t/2 m} \, d q$$

Assuming that the Gaussian is sufficiently narrow, we can extend the integration to $-\infty$ to ∞. The integrand is dominated where $q = k$, and we expand the exponent

$$\pm i q r - i \frac{\hbar q^2}{2 m} t = \pm i k r - i \frac{\hbar k^2}{2 m} t + i(\pm r - \frac{\hbar k}{m} t) (q - k) + O(q - k)^2.$$

We use the notation $v = \hbar k / m$, the classical velocity.

Within this approximation, the incoming piece is

$$\frac{d}{\sqrt{2\pi}} e^{-i k r - i \hbar k^2 t/2 m} \int_{-\infty}^\infty e^{-(q-k)^2 d^2/2} e^{i(-r - v t)(q-k)} \, d q$$

$$= e^{-i k r - i \hbar k^2 t/2 m} e^{-(r + v t)^2/2 d^2}$$

which is appreciable only if $t < 0$, while the outgoing piece is

$$\frac{d}{\sqrt{2\pi}} e^{i k r - i \hbar k^2 t/2 m} \int_{-\infty}^\infty e^{-(q-k)^2 d^2/2} e^{i(r - v t)(q-k)} \, d q$$

$$= e^{i k r - i \hbar k^2 t/2 m} e^{-(r - v t)^2/2 d^2}$$

which is appreciable only if $t > 0$.

(b)

Around the resonance, the phase shift is well approximated by $e^{2 i \delta_l(q)} = \frac{q-k_0-i\kappa}{q-k_0+i\kappa}$, and hence the scattered wave is

$$\frac{d}{\sqrt{2\pi}} \int_0^\infty e^{-(q-k)^2 d^2/2}\, e^{i q r}\, \frac{-2 i \kappa}{q-k_0+i\kappa}\, e^{-i \hbar q^2 t/2m}\, dq.$$

First of all, assuming that the Gaussian is wider than the resonance, we substitute k_0 into q, and we extend the integral from $-\infty$ to ∞,

$$\frac{d}{\sqrt{2\pi}} e^{-(k_0-k)^2 d^2/4} \int_{-\infty}^\infty e^{i q r}\, \frac{-2 i \kappa}{q-k_0+i\kappa}\, e^{-i \hbar q^2 t/2m}\, dq.$$

We expand the phase factor around k_0 up to the first order,

$$q r - \frac{\hbar q^2}{2m} t = k_0 r - \frac{\hbar k_0{}^2}{2m} t + (r - \frac{\hbar k_0}{m} t)(q-k_0) + O(q-k_0)^2,$$

$$\frac{d}{\sqrt{2\pi}} e^{-(k_0-k)^2 d^2/4}\, e^{i k_0 r - i \frac{\hbar k_0{}^2}{2m} t} \int_{-\infty}^\infty \frac{-2 i \kappa}{q-k_0+i\kappa}\, e^{i(r-\hbar k_0 t/m)(q-k_0)}\, dq$$

The integral can be extended to the lower half plane if $r - \frac{\hbar k_0}{m} t < 0$, and we find

$$\theta(v t - r) \frac{d}{\sqrt{2\pi}} e^{-(k_0-k)^2 d^2/2}\, e^{i k_0 r - i \frac{\hbar k_0{}^2}{2m} t} (-2\pi i)(-2 i \kappa)\, e^{i(r-\hbar k_0 t/m)(-i\kappa)}$$

$$= \theta(v t - r) \frac{d}{\sqrt{2\pi}} e^{-(k_0-k)^2 d^2/2}\, e^{i k_0 r - i \frac{\hbar k_0{}^2}{2m} t} (-4\pi\kappa)\, e^{\kappa r}\, e^{-\hbar k_0 \kappa t/m}.$$

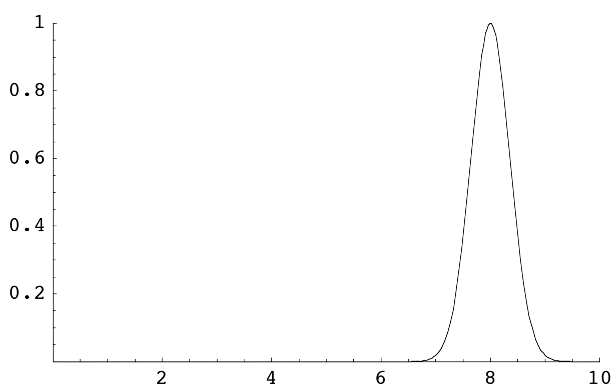
(c)

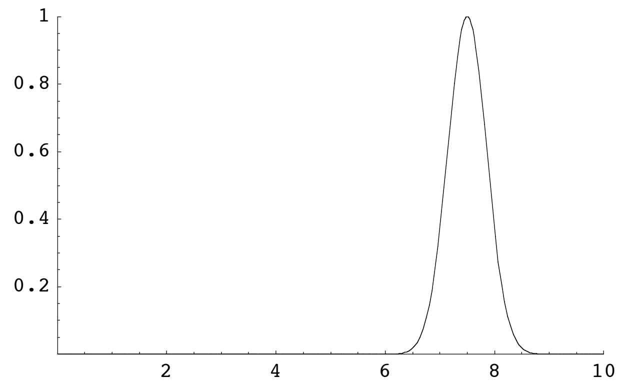
For plotting, I use $k = k_0 = 1$, $\kappa = 0.1$, $\hbar = 1$, $m = 1$, and $d = 3$.

```
rR = E^(ⅈ k r-ⅈ ħ k^2 t/(2 m)) E^(-(r-ħ k t/m)^2/(2 d^2)) - (-1)^l E^(-ⅈ k r-ⅈ ħ k^2 t/(2 m)) E^(-(r+ħ k t/m)^2/(2 d^2)) +
  If[r < ħ k0/m t, d/Sqrt[2 π] E^(-(k0-k)^2 d^2/2) E^(ⅈ k0 r-ⅈ ħ k0^2 t/(2 m)) (-4 π κ) E^(κ r) E^(-ħ k0 κ t/m), 0]
```

$$e^{i k r - \frac{i k^2 t \hbar}{2m} - \frac{\left(r - \frac{k t \hbar}{m}\right)^2}{2 d^2}} - (-1)^l e^{-i k r - \frac{i k^2 t \hbar}{2m} - \frac{\left(r + \frac{k t \hbar}{m}\right)^2}{2 d^2}} +$$
$$\mathrm{If}\left[r < \frac{t \hbar k_0}{m}, \frac{d\, e^{\frac{1}{2}(-(k_0-k)^2) d^2}\, e^{i k_0 r - \frac{i \hbar k_0^2 t}{2m}} (-4\pi\kappa)\, e^{\kappa r}\, e^{-\frac{\hbar k_0 \kappa t}{m}}}{\sqrt{2\pi}}, 0\right]$$

```
Table[Plot[Abs[rR]^2 /. {k → 1, k0 → 1, κ → 0.2, ħ → 1, m → 1, d → 0.5, l → 1},
  {r, 0, 10}, PlotRange → {{0, 10}, {0, 1}}], {t, -8, 8, 0.5}]
```

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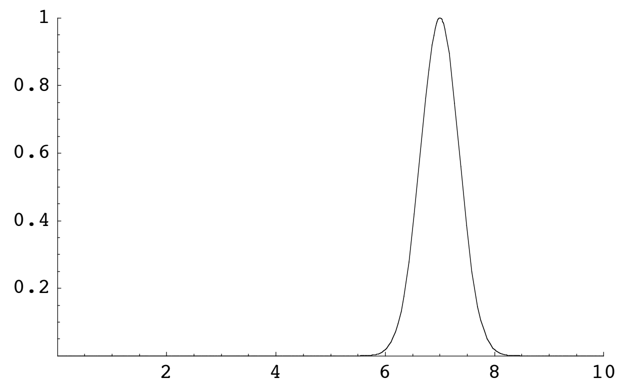
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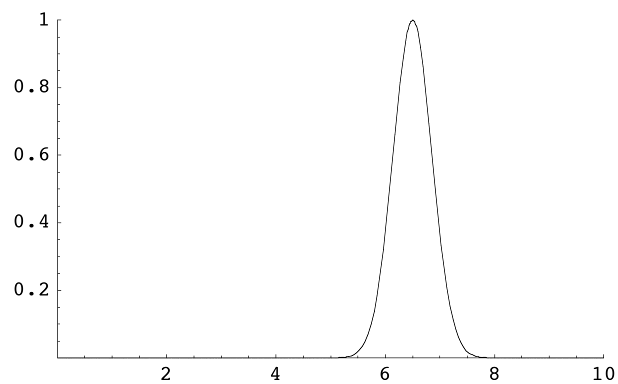
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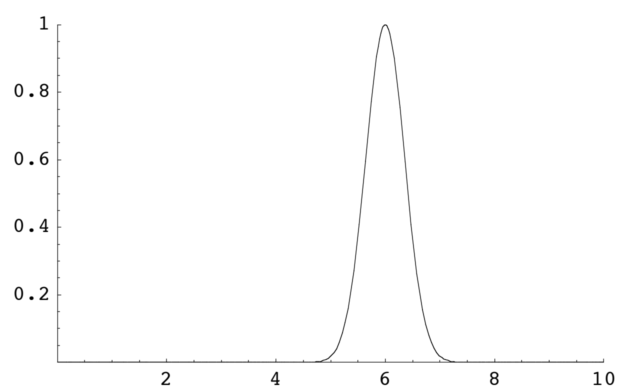
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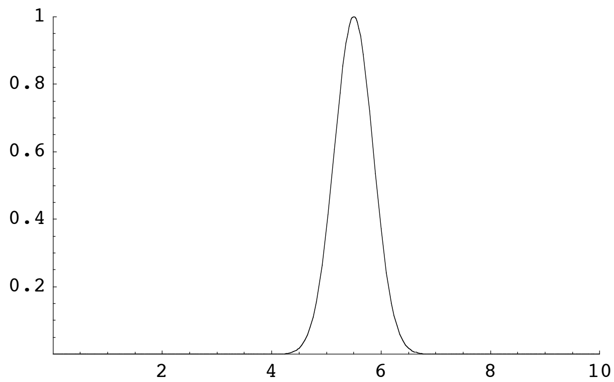
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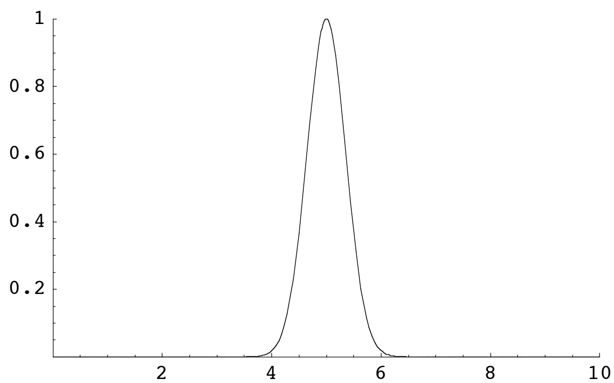
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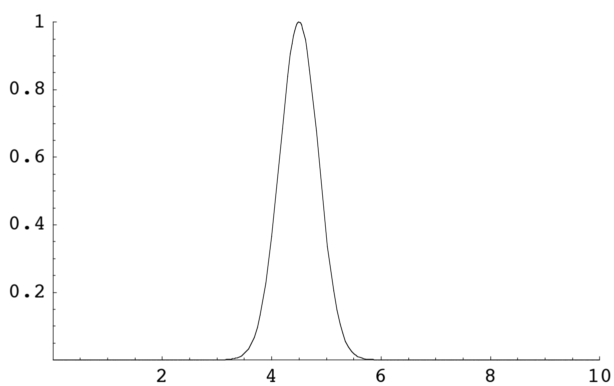
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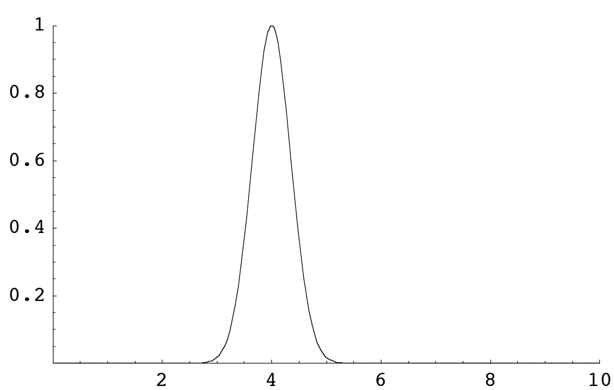
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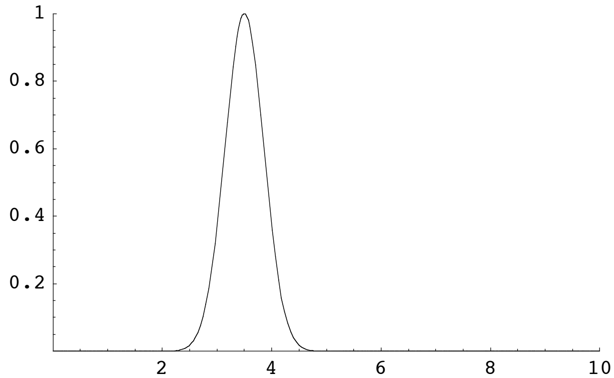
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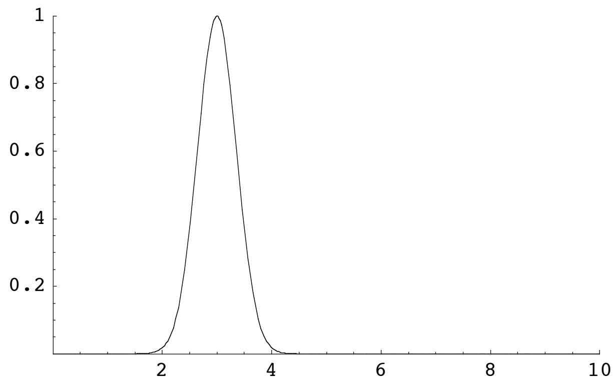
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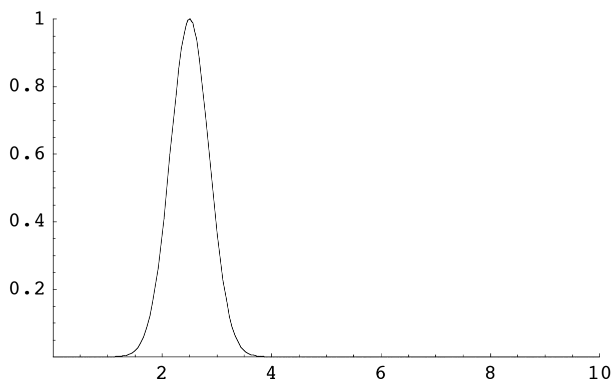
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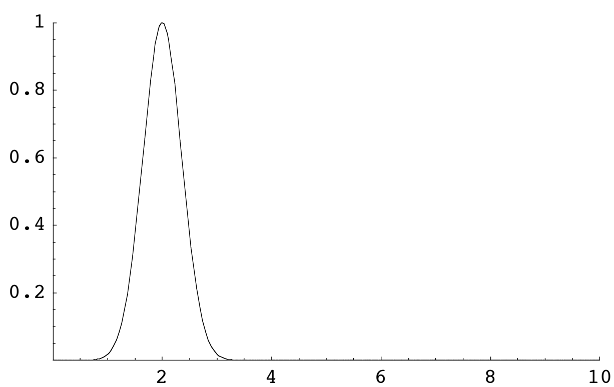
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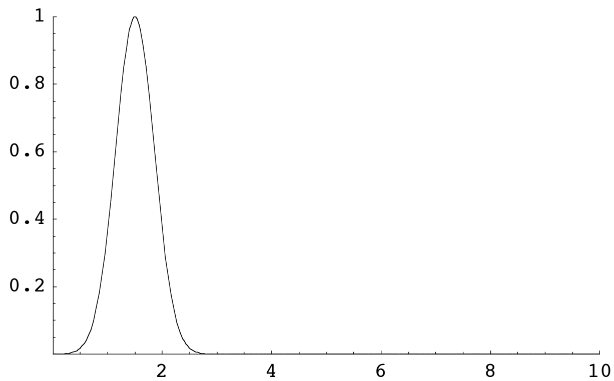
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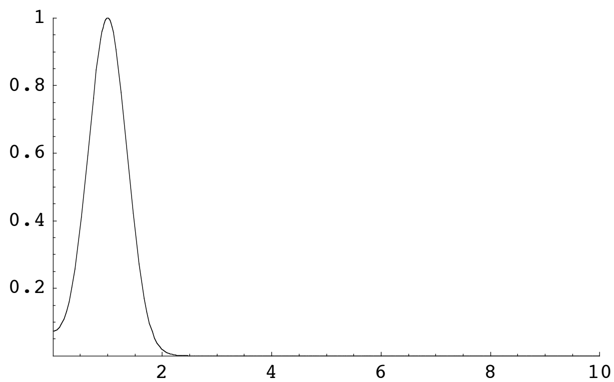
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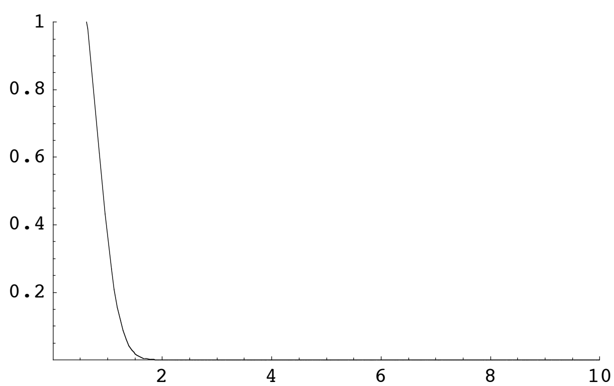
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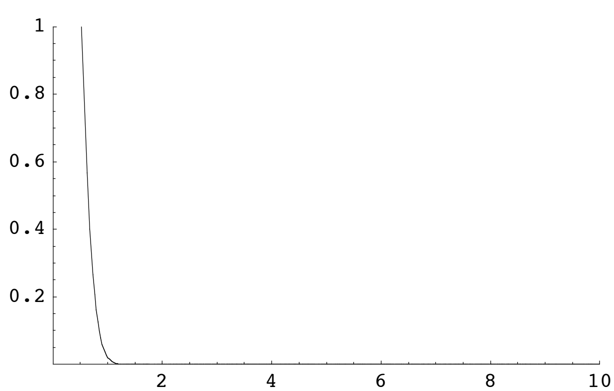
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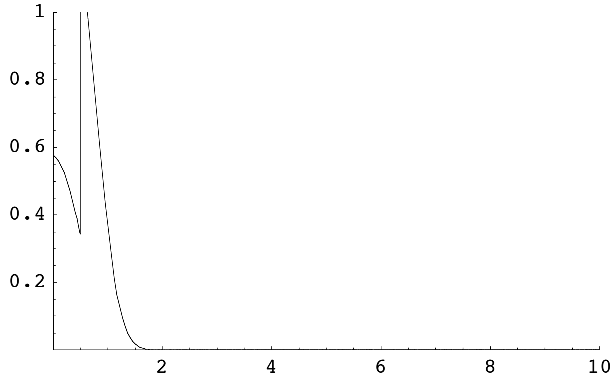
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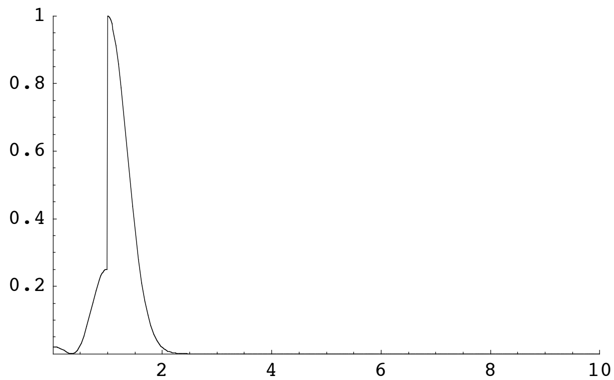
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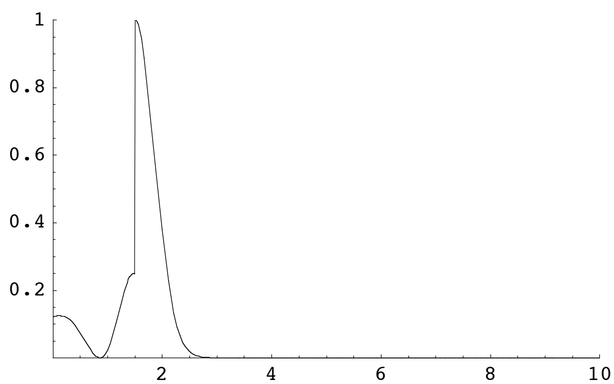
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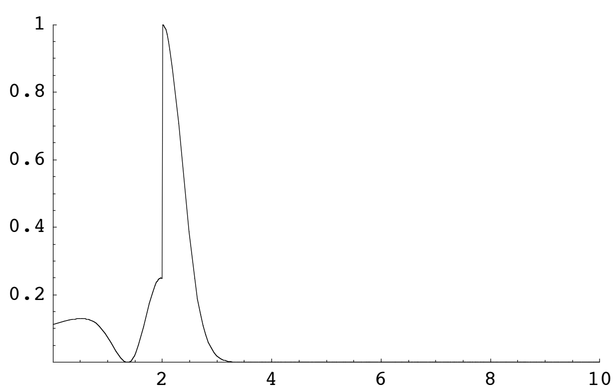
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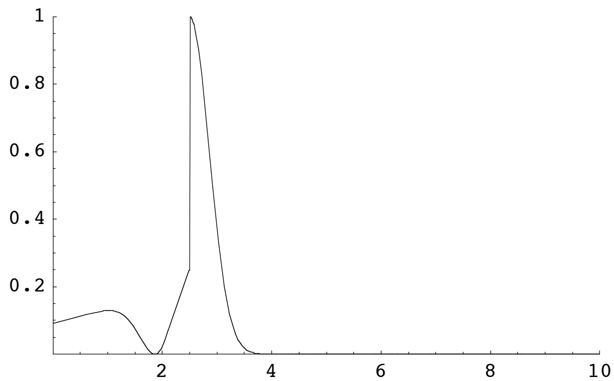
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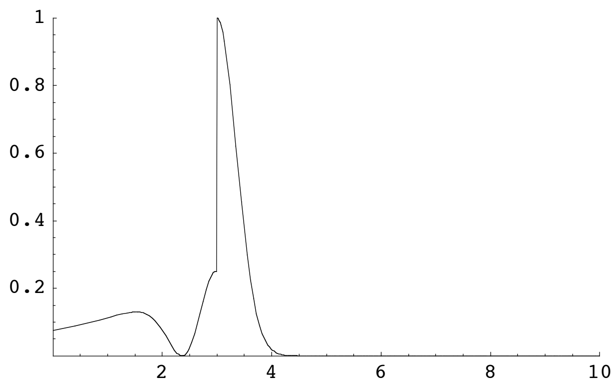
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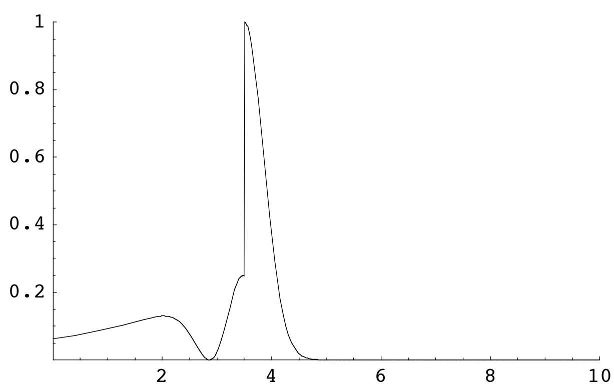
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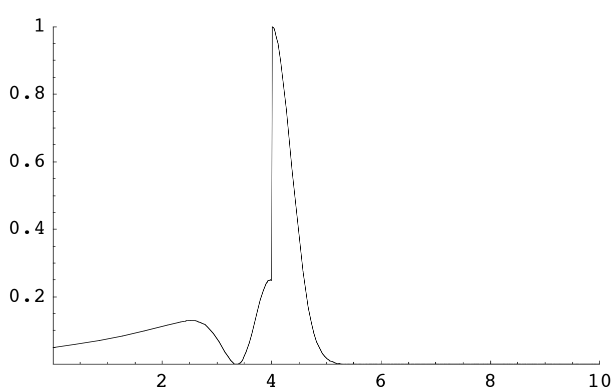
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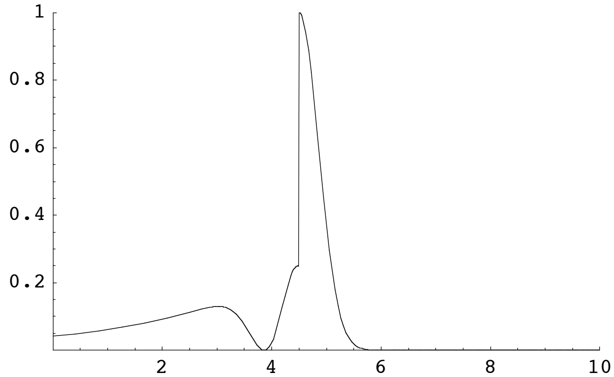
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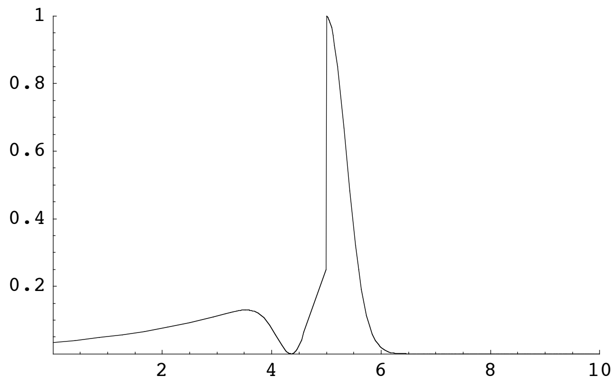
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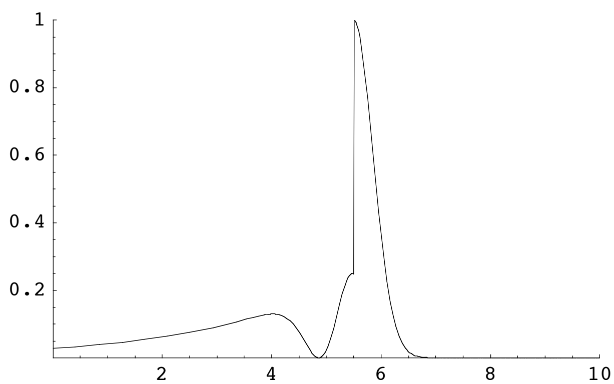
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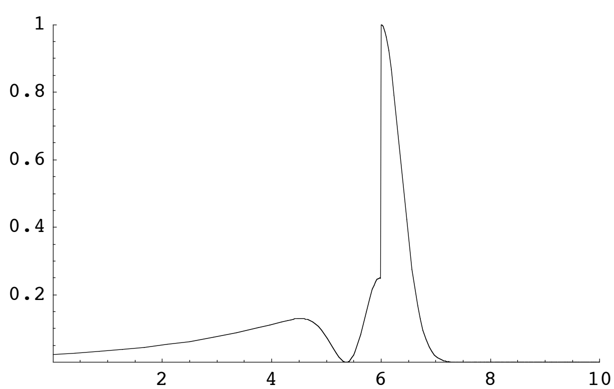
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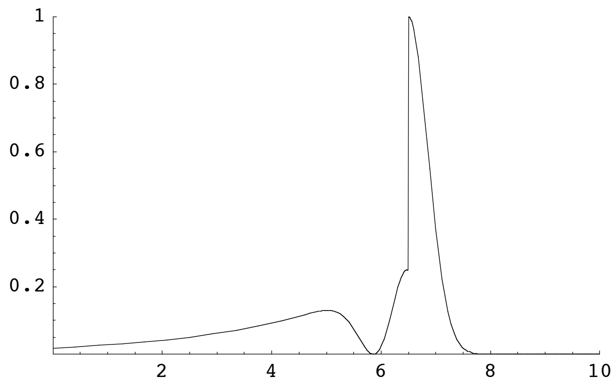
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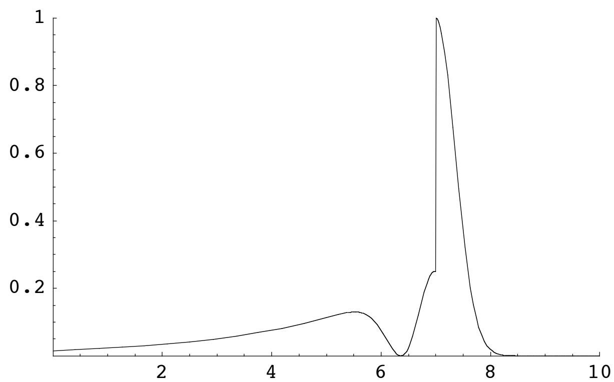
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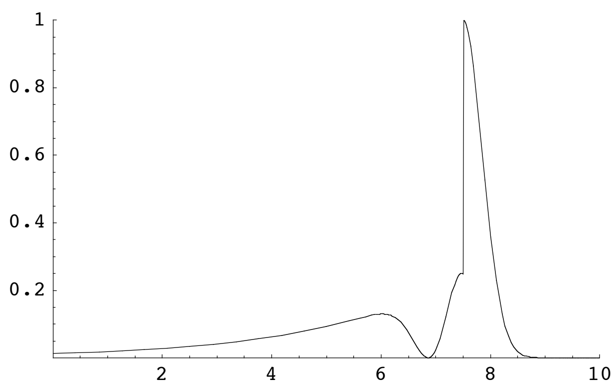
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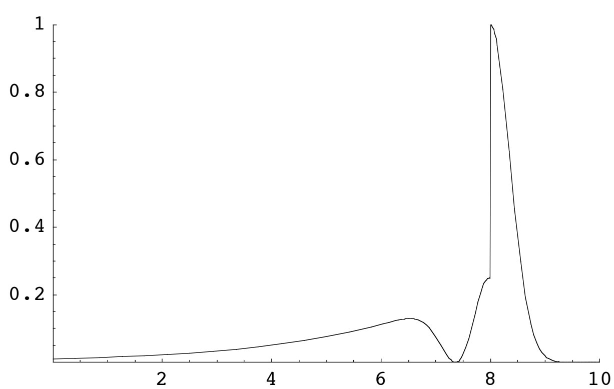
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I see the "delayed" piece with an exponential profile. It is also interesting to see that the probability is scraped off from the "prompt" Gaussian peak by a destructive interference, which is used to create the delayed piece consistent with the probability conservation, with a prominent dip right after the prompt peak.

(d)

The exponential tail has the time dependence $e^{-\hbar k_0 \kappa t/m} = e^{-t/2\tau}$ with $\tau = m/(2\hbar k_0 \kappa)$. Here, there is a factor of two in the exponent because the probability is the square of the wave function $e^{-t/\tau}$.

On the other hand, the imaginary part of the energy at the pole is $E = \frac{\hbar^2 (k_0 - i\kappa)^2}{2m} = \frac{\hbar^2 (k_0{}^2 - \kappa^2)}{2m} - i\frac{\hbar^2 k_0 \kappa}{m} = E_0 - i\frac{\Gamma}{2}$. Therefore, $\Gamma = \frac{2\hbar^2 k_0 \kappa}{m}$.

I find $\Gamma\tau = \hbar$.