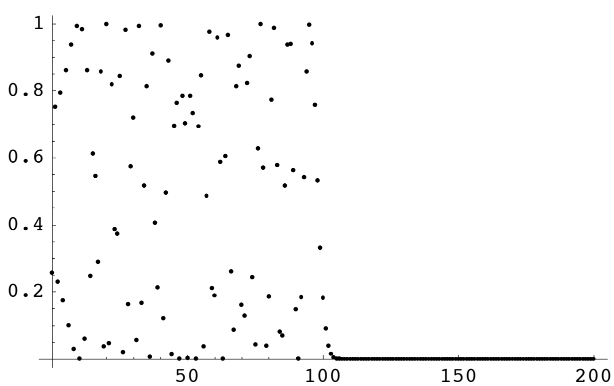
HW #4

1. Hard Sphere Scattering

For the hard sphere scattering, the requirement on the radial wave functionn is $R_l(r) = j_l(k\,r)\cos\delta_l + n_l(k\,r)\sin\delta_l$ for $r > a$, and $R_l(a) = 0$. Therefore, $\tan\delta_l = -\frac{j_l(k\,a)}{n_l(k\,a)}$. Using the trigonometric identity $\sin^2\theta = \frac{\tan^2\theta}{1+\tan^2\theta} = \frac{1}{1+\cos^2\theta}$, and for the case $k\,a = 100$,

```
sin2delta =
  Table[{l, N[(BesselJ[(2 l+1)/2, z])^2 / ((BesselJ[(2 l+1)/2, z])^2 + (BesselY[(2 l+1)/2, z])^2) /. {z → 100}]}, {l, 0, 200}];
```

```
ListPlot[sin2delta]
```

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- Graphics -
```

It appears that $\sin^2\delta_l$ behaves more or less randomly between 0 and 1, and hence $\frac{1}{2}$ on average.

```
Sum[4 π (2 l + 1)/100^2 sin2delta[[l + 1, 2]], {l, 0, 200}]
```

```
6.56891
```

We can analytically estimate the total cross section in the $k\,a \gg 1$ limit, by regarding $\sin^2\delta_l = \frac{1}{2}$ on average for $0 \le l \le k\,a$. Then $\sigma = \sum_{l=0}^{k\,a} \frac{4\pi(2\,l+1)}{k^2}\,\frac{1}{2} = \frac{2\pi}{k^2}(k\,a+1)^2 = 2\pi a^2$ up to corrections of order $(k\,a)^{-1}$. Indeed, the total cross section above for $k\,a = 100$ is quite close to

```
N[2 π a^2 /. {a → 1}]
```

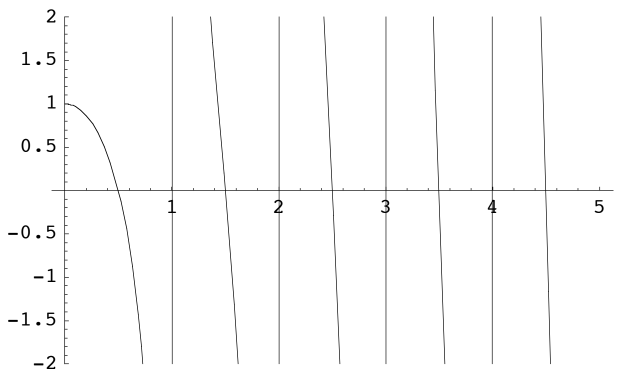
```
6.28319
```

2. Spherical Well Potential for $l = 0$. (Threshold bound states)

(a)

Bound state problem is given by $r R_0(r) = \sin\sqrt{K^2 - \kappa^2}\, r$ for $r < a$, and $r R_0(r) = e^{-\kappa r}$ for $r > a$. The continuity of the logarithmic derivatives requires $\sqrt{K^2 - \kappa^2} \cot\sqrt{K^2 - \kappa^2}\, a = -\kappa$ or $\frac{\kappa}{\sqrt{K^2 - \kappa^2}} \tan\sqrt{K^2 - \kappa^2}\, a + 1 = 0$. To understand where the solutions appear, we plot the function $z \cot z$

```
Plot[π x Cot[π x], {x, 0, 5}, PlotRange → {-2, 2}]
```



```
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```

Note that the straight vertical lines are an artefact of the plotting by *Mathematica*.

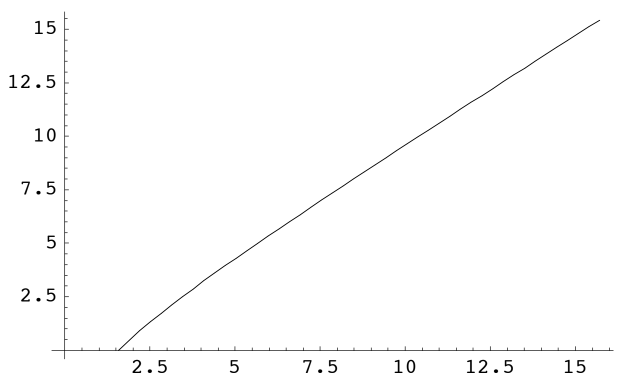
For a bound state to form, we need $\sqrt{K^2 - \kappa^2} \cot\sqrt{K^2 - \kappa^2}\, a$ to be the same as $-\kappa < 0$. Therefore, the solutions arise when $(n + \frac{1}{2})\pi < \sqrt{K^2 - \kappa^2}\, a < (n + 1)\pi$, or $\sqrt{K^2 - ((n + 1)\pi)^2} < \kappa < \sqrt{K^2 - ((n + \frac{1}{2})\pi)^2}$.
For the threshold bound states, $\kappa = 0$ and hence $K \cot K a = 0$, and therefore $K a = (n + \frac{1}{2})\pi$. New solutions appear there, and the binding energies increase as K is increased.

The rest is not required, but this is how one can plot the binding energies as a function of the potential depth.

sol1 = Table$\left[\left\{K, \kappa \mathbin{/.} \text{FindRoot}\left[\sqrt{K^2-\kappa^2}\ \text{Cot}\left[\sqrt{K^2-\kappa^2}\ a\right] + \kappa \mathbin{/.} \{a \to 1\},\right.\right.\right.$
$\left.\left.\left.\left\{\kappa, \sqrt{K^2-\left(\frac{1}{2}\pi\right)^2}, \sqrt{\text{Max}[K^2-\pi^2, 0]}, \sqrt{K^2-\left(\frac{1}{2}\pi\right)^2}\right\}\right][[1]]\right\}, \left\{K, \frac{1}{2}\pi, 5\pi, \frac{1}{10}\pi\right\}\right]$

$\{\{\frac{\pi}{2}, 0.\}, \{\frac{3\pi}{5}, 0.473252\}, \{\frac{7\pi}{10}, 0.914789\}, \{\frac{4\pi}{5}, 1.33373\}, \{\frac{9\pi}{10}, 1.73588\}, \{\pi, 2.12514\},$
$\{\frac{11\pi}{10}, 2.50428\}, \{\frac{6\pi}{5}, 2.87526\}, \{\frac{13\pi}{10}, 3.2396\}, \{\frac{7\pi}{5}, 3.59842\}, \{\frac{3\pi}{2}, 3.95262\},$
$\{\frac{8\pi}{5}, 4.30288\}, \{\frac{17\pi}{10}, 4.64979\}, \{\frac{9\pi}{5}, 4.99379\}, \{\frac{19\pi}{10}, 5.33527\}, \{2\pi, 5.67453\},$
$\{\frac{21\pi}{10}, 6.01184\}, \{\frac{11\pi}{5}, 6.34743\}, \{\frac{23\pi}{10}, 6.68148\}, \{\frac{12\pi}{5}, 7.01415\}, \{\frac{5\pi}{2}, 7.34559\},$
$\{\frac{13\pi}{5}, 7.67591\}, \{\frac{27\pi}{10}, 8.00522\}, \{\frac{14\pi}{5}, 8.33362\}, \{\frac{29\pi}{10}, 8.66118\}, \{3\pi, 8.98798\},$
$\{\frac{31\pi}{10}, 9.31408\}, \{\frac{16\pi}{5}, 9.63954\}, \{\frac{33\pi}{10}, 9.96441\}, \{\frac{17\pi}{5}, 10.2887\}, \{\frac{7\pi}{2}, 10.6126\},$
$\{\frac{18\pi}{5}, 10.9359\}, \{\frac{37\pi}{10}, 11.2588\}, \{\frac{19\pi}{5}, 11.5813\}, \{\frac{39\pi}{10}, 11.9035\}, \{4\pi, 12.2253\},$
$\{\frac{41\pi}{10}, 12.5467\}, \{\frac{21\pi}{5}, 12.8679\}, \{\frac{43\pi}{10}, 13.1887\}, \{\frac{22\pi}{5}, 13.5093\}, \{\frac{9\pi}{2}, 13.8296\},$
$\{\frac{23\pi}{5}, 14.1497\}, \{\frac{47\pi}{10}, 14.4696\}, \{\frac{24\pi}{5}, 14.7893\}, \{\frac{49\pi}{10}, 15.1087\}, \{5\pi, 15.428\}\}$

plot1 = ListPlot[sol1, PlotJoined → True]

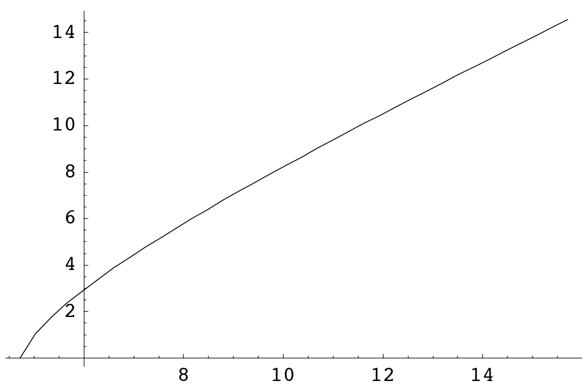


- Graphics -

$$\textbf{sol2} = \textbf{Table}\left[\left\{K, \varkappa \text{ /. FindRoot}\left[\sqrt{K^2 - \varkappa^2}\ \text{Cot}\left[\sqrt{K^2 - \varkappa^2}\ a\right] + \varkappa \text{ /. } \{a \to 1\}, \left\{\varkappa, \sqrt{K^2 - \left(\frac{3}{2}\pi\right)^2}, \sqrt{\text{Max}[K^2 - (2\pi)^2, 0]}, \sqrt{K^2 - \left(\frac{3}{2}\pi\right)^2}\right\}\right][[1]]\right\}, \left\{K, \frac{3}{2}\pi, 5\pi, \frac{1}{10}\pi\right\}\right]$$

$$\left\{\left\{\frac{3\pi}{2}, 0.\right\}, \left\{\frac{8\pi}{5}, 1.03272\right\}, \left\{\frac{17\pi}{10}, 1.74932\right\}, \left\{\frac{9\pi}{5}, 2.35381\right\}, \left\{\frac{19\pi}{10}, 2.89665\right\}, \{2\pi, 3.39955\}, \left\{\frac{21\pi}{10}, 3.87416\right\}, \left\{\frac{11\pi}{5}, 4.32756\right\}, \left\{\frac{23\pi}{10}, 4.7644\right\}, \left\{\frac{12\pi}{5}, 5.18794\right\}, \left\{\frac{5\pi}{2}, 5.60053\right\}, \left\{\frac{13\pi}{5}, 6.00397\right\}, \left\{\frac{27\pi}{10}, 6.39963\right\}, \left\{\frac{14\pi}{5}, 6.78861\right\}, \left\{\frac{29\pi}{10}, 7.17178\right\}, \{3\pi, 7.54986\}, \left\{\frac{31\pi}{10}, 7.92344\right\}, \left\{\frac{16\pi}{5}, 8.29304\right\}, \left\{\frac{33\pi}{10}, 8.65906\right\}, \left\{\frac{17\pi}{5}, 9.02187\right\}, \left\{\frac{7\pi}{2}, 9.38178\right\}, \left\{\frac{18\pi}{5}, 9.73905\right\}, \left\{\frac{37\pi}{10}, 10.0939\right\}, \left\{\frac{19\pi}{5}, 10.4466\right\}, \left\{\frac{39\pi}{10}, 10.7973\right\}, \{4\pi, 11.1462\}, \left\{\frac{41\pi}{10}, 11.4933\right\}, \left\{\frac{21\pi}{5}, 11.8389\right\}, \left\{\frac{43\pi}{10}, 12.183\right\}, \left\{\frac{22\pi}{5}, 12.5257\right\}, \left\{\frac{9\pi}{2}, 12.8672\right\}, \left\{\frac{23\pi}{5}, 13.2076\right\}, \left\{\frac{47\pi}{10}, 13.5468\right\}, \left\{\frac{24\pi}{5}, 13.885\right\}, \left\{\frac{49\pi}{10}, 14.2222\right\}, \{5\pi, 14.5585\}\right\}$$

plot2 = ListPlot[sol2, PlotJoined → True]

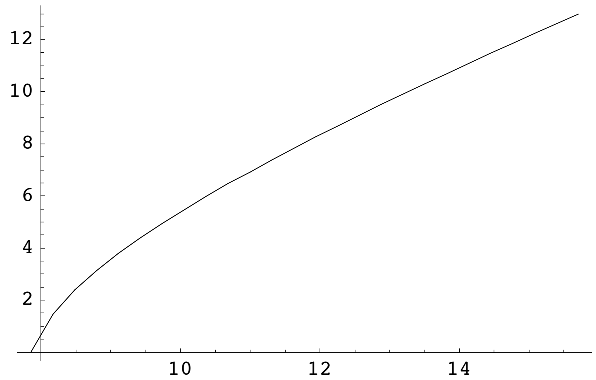


- Graphics -

$$\textbf{sol3} = \textbf{Table}\left[\left\{K, \varkappa \text{ /. FindRoot}\left[\sqrt{K^2 - \varkappa^2}\ \text{Cot}\left[\sqrt{K^2 - \varkappa^2}\ a\right] + \varkappa \text{ /. } \{a \to 1\}, \left\{\varkappa, \sqrt{K^2 - \left(\frac{5}{2}\pi\right)^2}, \sqrt{\text{Max}[K^2 - (3\pi)^2, 0]}, \sqrt{K^2 - \left(\frac{5}{2}\pi\right)^2}\right\}\right][[1]]\right\}, \left\{K, \frac{5}{2}\pi, 5\pi, \frac{1}{10}\pi\right\}\right]$$

$$\left\{\left\{\frac{5\pi}{2}, 0.\right\}, \left\{\frac{13\pi}{5}, 1.46948\right\}, \left\{\frac{27\pi}{10}, 2.38764\right\}, \left\{\frac{14\pi}{5}, 3.13592\right\}, \left\{\frac{29\pi}{10}, 3.79322\right\}, \{3\pi, 4.39218\}, \left\{\frac{31\pi}{10}, 4.95\right\}, \left\{\frac{16\pi}{5}, 5.477\right\}, \left\{\frac{33\pi}{10}, 5.97992\right\}, \left\{\frac{17\pi}{5}, 6.46347\right\}, \left\{\frac{7\pi}{2}, 6.93106\right\}, \left\{\frac{18\pi}{5}, 7.38525\right\}, \left\{\frac{37\pi}{10}, 7.82803\right\}, \left\{\frac{19\pi}{5}, 8.26096\right\}, \left\{\frac{39\pi}{10}, 8.6853\right\}, \{4\pi, 9.1021\}, \left\{\frac{41\pi}{10}, 9.5122\right\}, \left\{\frac{21\pi}{5}, 9.91633\right\}, \left\{\frac{43\pi}{10}, 10.3151\right\}, \left\{\frac{22\pi}{5}, 10.709\right\}, \left\{\frac{9\pi}{2}, 11.0986\right\}, \left\{\frac{23\pi}{5}, 11.4841\right\}, \left\{\frac{47\pi}{10}, 11.866\right\}, \left\{\frac{24\pi}{5}, 12.2445\right\}, \left\{\frac{49\pi}{10}, 12.62\right\}, \{5\pi, 12.9926\}\right\}$$

plot3 = ListPlot[sol3, PlotJoined → True]



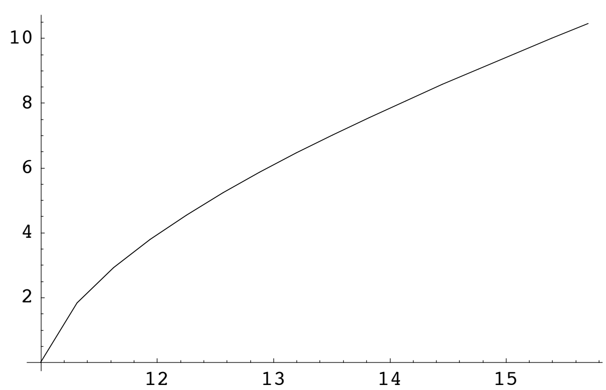
- Graphics -

sol4 = Table$\left[\left\{K, \kappa \,/.\, \text{FindRoot}\left[\sqrt{K^2-\kappa^2}\ \text{Cot}\left[\sqrt{K^2-\kappa^2}\ a\right] + \kappa \,/.\, \{a \to 1\},\ \left\{\kappa, \sqrt{K^2-\left(\frac{7}{2}\pi\right)^2}, \sqrt{\text{Max}[K^2-(4\pi)^2, 0]}, \sqrt{K^2-\left(\frac{7}{2}\pi\right)^2}\right\}\right][[1]]\right\}, \left\{K, \frac{7}{2}\pi, 5\pi, \frac{1}{10}\pi\right\}\right]$

FindRoot::reged : The point {0.} is at the edge of the search region {0., 0.} in coordinate 1 and the computed search direction points outside the region. More…

$\left\{\left\{\frac{7\pi}{2}, 0.\right\}, \left\{\frac{18\pi}{5}, 1.84024\right\}, \left\{\frac{37\pi}{10}, 2.92474\right\}, \left\{\frac{19\pi}{5}, 3.79432\right\}, \left\{\frac{39\pi}{10}, 4.54997\right\}, \{4\pi, 5.23274\}, \left\{\frac{41\pi}{10}, 5.86408\right\}, \left\{\frac{21\pi}{5}, 6.45683\right\}, \left\{\frac{43\pi}{10}, 7.01939\right\}, \left\{\frac{22\pi}{5}, 7.55759\right\}, \left\{\frac{9\pi}{2}, 8.07566\right\}, \left\{\frac{23\pi}{5}, 8.57679\right\}, \left\{\frac{47\pi}{10}, 9.06345\right\}, \left\{\frac{24\pi}{5}, 9.53759\right\}, \left\{\frac{49\pi}{10}, 10.0008\right\}, \{5\pi, 10.4543\}\right\}$

plot4 = ListPlot[sol4, PlotJoined → True]



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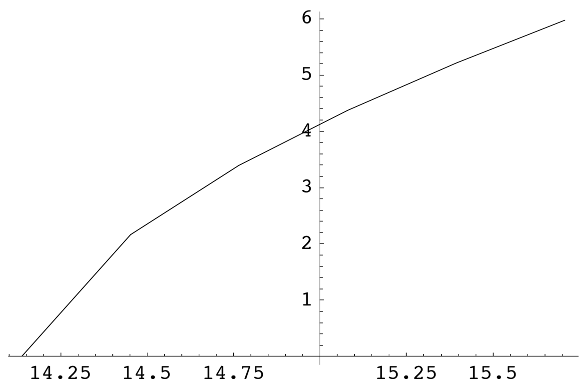
```
sol5 = Table[{K, κ /. FindRoot[√(K² - κ²) Cot[√(K² - κ²) a] + κ /. {a → 1},
      {κ, √(K² - (9/2 π)²), √(Max[K² - (5 π)², 0]), √(K² - (9/2 π)²)}][[1]]}, {K, 9/2 π, 5 π, 1/10 π}]
```

```
FindRoot::reged : The point {0.} is at the edge of the search region {0., 0.}
   in coordinate 1 and the computed search direction points outside the region. More…
```

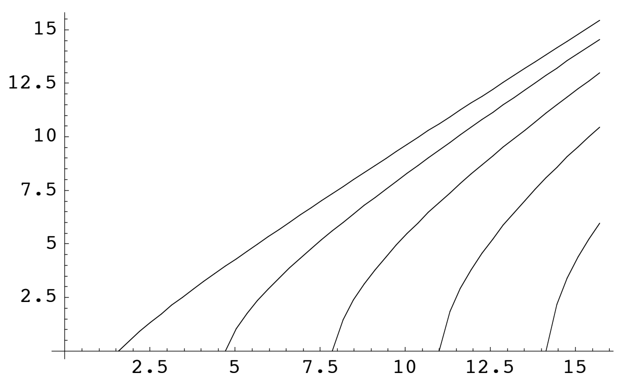
$\{\{\frac{9\pi}{2}, 0.\}, \{\frac{23\pi}{5}, 2.1681\}, \{\frac{47\pi}{10}, 3.39733\}, \{\frac{24\pi}{5}, 4.37371\}, \{\frac{49\pi}{10}, 5.21678\}, \{5\pi, 5.97463\}\}$

```
plot5 = ListPlot[sol5, PlotJoined → True]
```



- Graphics -

```
Show[plot1, plot2, plot3, plot4, plot5]
```



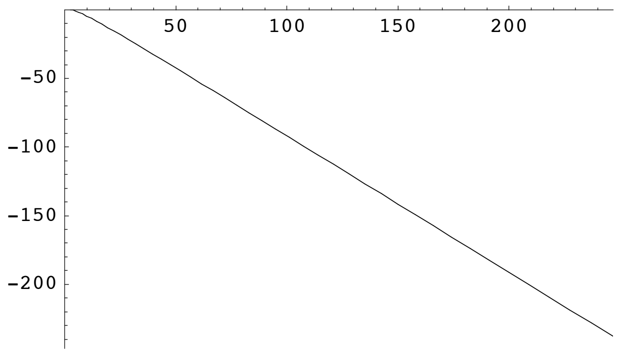
- Graphics -

This plot shows the relationship between K on the horizontal axis and κ on the vertical axis. New bound states indeed appear at $K = (n + \frac{1}{2})\,\frac{\pi}{a}$. To plot the bound state energies $E = -\hbar^2 \kappa^2 / 2m$ as a function of $V_0 = 2 m K^2 / \hbar^2$, we need to just square them in the unit of $2m/\hbar^2 = 1$.

```
Dimensions[sol1]
```

{46, 2}

```
E1 = ListPlot[Table[{sol1[[i, 1]]^2, -sol1[[i, 2]]^2}, {i, 1, 46}],
  PlotJoined → True, PlotRange → {{0, (5 π)^2}, {-(5 π)^2, 0}}]
```

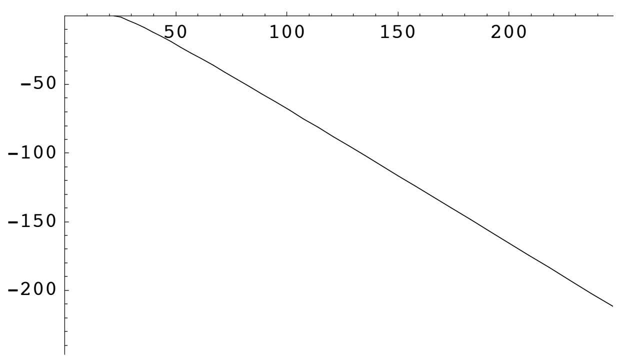


- Graphics -

```
Dimensions[sol2]
```

{36, 2}

```
E2 = ListPlot[Table[{sol2[[i, 1]]^2, -sol2[[i, 2]]^2}, {i, 1, 36}],
  PlotJoined → True, PlotRange → {{0, (5 π)^2}, {-(5 π)^2, 0}}]
```

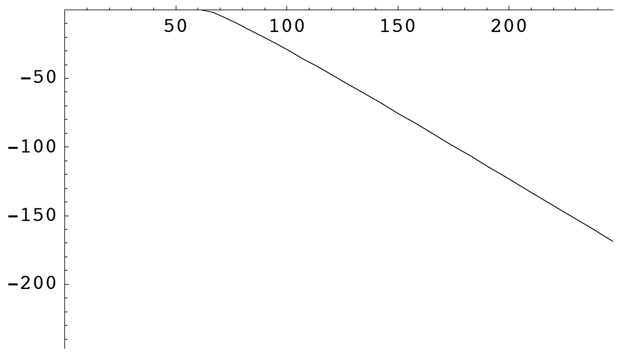


- Graphics -

```
Dimensions[sol3]
```

{26, 2}

```
E3 = ListPlot[Table[{sol3[[i, 1]]^2, -sol3[[i, 2]]^2}, {i, 1, 26}],
  PlotJoined → True, PlotRange → {{0, (5 π)^2}, {-(5 π)^2, 0}}]
```

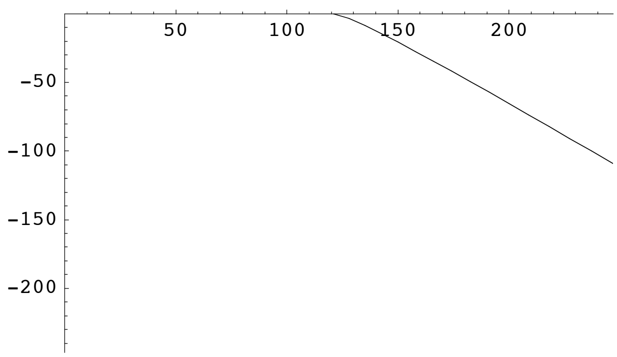


- Graphics -

```
Dimensions[sol4]
```

{16, 2}

```
E4 = ListPlot[Table[{sol4[[i, 1]]^2, -sol4[[i, 2]]^2}, {i, 1, 16}],
  PlotJoined → True, PlotRange → {{0, (5 π)^2}, {-(5 π)^2, 0}}]
```



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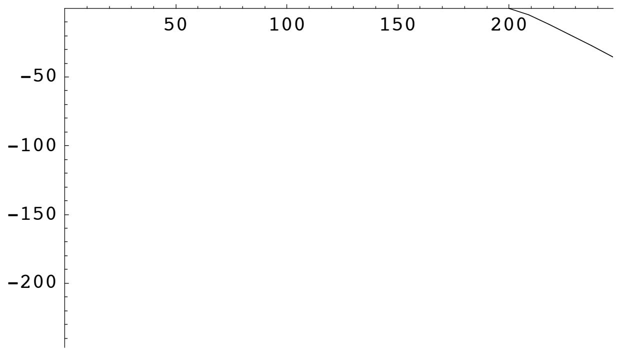
```
Dimensions[sol5]
```

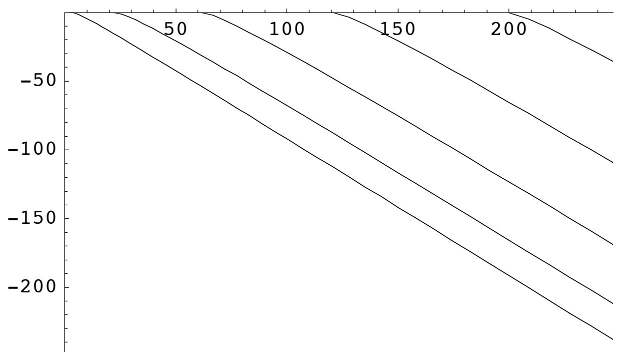
{6, 2}

```
E5 = ListPlot[Table[{sol5[[i, 1]]^2, -sol5[[i, 2]]^2}, {i, 1, 6}],
  PlotJoined → True, PlotRange → {{0, (5 π)^2}, {-(5 π)^2, 0}}]
```



- Graphics -

```
Show[E1, E2, E3, E4, E5]
```



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This plot shows how new bound states appear at each $V_0 = \frac{1}{2 m a^2} \hbar^2 ((n + \frac{1}{2}) \pi)^2$ as the depth of the potential is increased.

(b)

Phase shift is given by matching $r\,R_0(r) = \sin\sqrt{K^2+k^2}\;r$ for $r < a$ and $r\,R_0(r) = \sin k\,r\cos\delta_0 + \cos k\,r\sin\delta_0 = \sin(k\,r+\delta_0)$. The logarithmic derivatives are $\sqrt{K^2+k^2}\,\cot\sqrt{K^2+k^2}\;a = k\cot(k\,a-\delta_0)$, or $\frac{k}{\sqrt{K^2+k^2}}\tan\sqrt{K^2+k^2}\;a = \tan(k\,a+\delta_0)$. therefore, $\delta_0 = -k\,a + \arctan\frac{k}{\sqrt{K^2+k^2}}\tan\sqrt{K^2+k^2}\;a$.

We can rewrite it as

$$\frac{k}{\sqrt{K^2+k^2}}\tan\sqrt{K^2+k^2}\;a = \tan(k\,a+\delta_0)$$

$$= -i\,\frac{e^{i(k\,a+\delta_0)}-e^{-i(k\,a+\delta_0)}}{e^{i(k\,a+\delta_0)}+e^{-i(k\,a+\delta_0)}} = -i\,\frac{e^{i\,k\,a}\,e^{2\,i\,\delta_0}-e^{-i\,k\,a}}{e^{i\,k\,a}\,e^{2\,i\,\delta_0}+e^{-i\,k\,a}}$$

we find

$$i\,\frac{k}{\sqrt{K^2+k^2}}\tan\sqrt{K^2+k^2}\;a(e^{i\,k\,a}\,e^{2\,i\,\delta_0}+e^{-i\,k\,a}) = e^{i\,k\,a}\,e^{2\,i\,\delta_0}-e^{-i\,k\,a}$$

and

$$e^{2\,i\,\delta_0}\,e^{i\,k\,a}\left(i\,\frac{k}{\sqrt{K^2+k^2}}\tan\sqrt{K^2+k^2}\;a-1\right) = \left(-i\,\frac{k}{\sqrt{K^2+k^2}}\tan\sqrt{K^2+k^2}\;a-1\right)$$

and hence

$$e^{2\,i\,\delta_0} = e^{-i\,k\,a}\,\frac{1+i\,\frac{k}{\sqrt{K^2+k^2}}\tan\sqrt{K^2+k^2}\;a}{1-i\,\frac{k}{\sqrt{K^2+k^2}}\tan\sqrt{K^2+k^2}\;a}$$

(c)

The S-matrix element $e^{2\,i\,\delta_0}$ has poles when

$$1-i\,\frac{k}{\sqrt{K^2+k^2}}\tan\sqrt{K^2+k^2}\;a = 0\,.$$

On the upper half plane, we can write $k = i\,\kappa$, where Re(κ)>0. The poles appear where

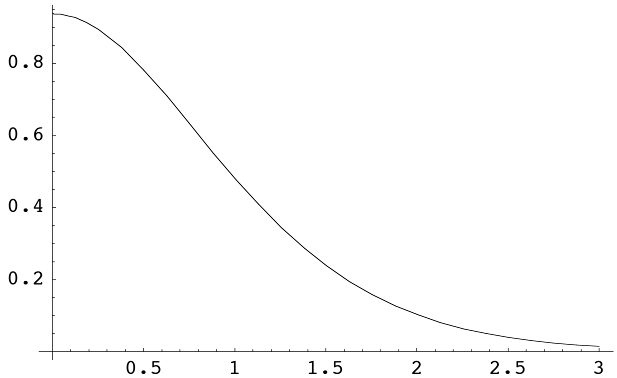
$1+\frac{\kappa}{\sqrt{K^2-\kappa^2}}\tan\sqrt{K^2-\kappa^2}\;a = 0$. This is precisely the condition for the bound states studied in part (a).

```
δ0 = ArcTan[k/Sqrt[k^2 + K^2] Tan[Sqrt[k^2 + K^2] a]] - k a
```

```
-a k + ArcTan[k Tan[a Sqrt[k^2 + K^2]]/Sqrt[k^2 + K^2]]
```

Far away from a threshold bound state, for instance for $K = \frac{\pi}{4}$,

```
Plot[4π/k² Sin[δ₀]² /. {a → 1, K → π/4}, {k, 0, 3}]
```



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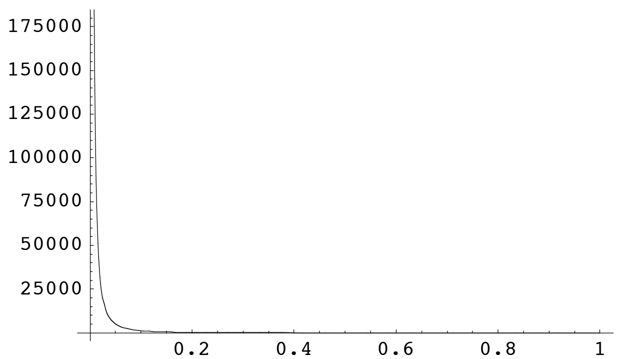
which is smaller than the geometric cross section. On the other hand, exactly on the threshold resonance,

```
Plot[4π/k² Sin[δ₀]² /. {a → 1, K → π/2}, {k, 0, 1}]
```



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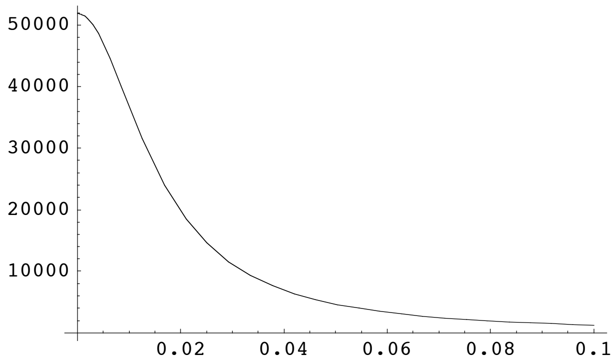
and the cross section diverges at $k = 0$. For K just above the threshold bound state, namely when the bound state exists just below $E = 0$,

```
Plot[4π/k² Sin[δ₀]² /. {a → 1, K → π/2 + 0.01}, {k, 0, 0.1}]
```



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The cross section is finite, but is much bigger than the geometric cross section. Similarly, for K just below the threshold bound state, namely when the bound state has just disappeared above $E = 0$,

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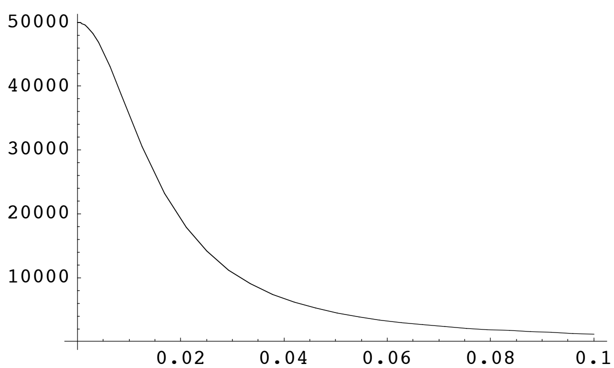
```
Plot[4π/k² Sin[δ₀]² /. {a → 1, K → π/2 - 0.01}, {k, 0, 0.1}]
```



- Graphics -

It still exhibits a cross section much larger than the geometric cross section.

3. Spherical Well Potential for $l = 1$. (Resonances)

(a)

Here is $j_1(z)$:

```
PowerExpand[Sqrt[π/(2 z)] BesselJ[3/2, z]]
```

$$\frac{-\text{Cos}[z]+\frac{\text{Sin}[z]}{z}}{z}$$

Here is $n_1(z)$:

```
PowerExpand[Sqrt[π/(2 z)] BesselY[3/2, z]]
```

$$\frac{-\frac{\text{Cos}[z]}{z}-\text{Sin}[z]}{z}$$

We use $r\,R_1(r)$

```
Rin = -Cos[Sqrt[k^2 + K^2] r] + Sin[Sqrt[k^2 + K^2] r]/(Sqrt[k^2 + K^2] r)
```

$$-\text{Cos}\left[\sqrt{k^2+K^2}\;r\right]+\frac{\text{Sin}\left[\sqrt{k^2+K^2}\;r\right]}{\sqrt{k^2+K^2}\;r}$$

```
Rout = (-Cos[k r] + Sin[k r]/(k r)) Cos[δ1] + (-Cos[k r]/(k r) - Sin[k r]) Sin[δ1]
```

General::spell1 : Possible spelling error: new symbol name "Rout" is similar to existing symbol "Root". More…

$$\text{Cos}[\delta_1]\left(-\text{Cos}[k\,r]+\frac{\text{Sin}[k\,r]}{k\,r}\right)+\left(-\frac{\text{Cos}[k\,r]}{k\,r}-\text{Sin}[k\,r]\right)\text{Sin}[\delta_1]$$

```
Simplify[Rout]
```

$$-\text{Cos}[k\,r-\delta_1]+\frac{\text{Sin}[k\,r-\delta_1]}{k\,r}$$

```
Simplify[D[Rin, r]/Rin]
```

$$-\left(\sqrt{k^2+K^2}\;r\,\text{Cos}\left[\sqrt{k^2+K^2}\;r\right]+\left(-1+k^2 r^2+K^2 r^2\right)\text{Sin}\left[\sqrt{k^2+K^2}\;r\right]\right)\Big/\left(r\left(\sqrt{k^2+K^2}\;r\,\text{Cos}\left[\sqrt{k^2+K^2}\;r\right]-\text{Sin}\left[\sqrt{k^2+K^2}\;r\right]\right)\right)$$

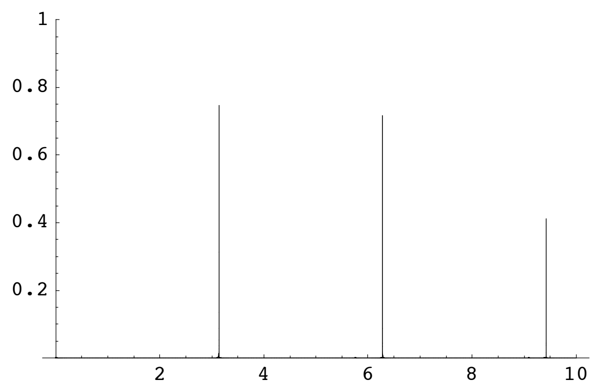
```
D[Rout, r]/Rout
```

$$\left(\text{Cos}[\delta_1]\left(\frac{\text{Cos}[k\,r]}{r}+k\,\text{Sin}[k\,r]-\frac{\text{Sin}[k\,r]}{k\,r^2}\right)+\left(-k\,\text{Cos}[k\,r]+\frac{\text{Cos}[k\,r]}{k\,r^2}+\frac{\text{Sin}[k\,r]}{r}\right)\text{Sin}[\delta_1]\right)\Big/\left(\text{Cos}[\delta_1]\left(-\text{Cos}[k\,r]+\frac{\text{Sin}[k\,r]}{k\,r}\right)+\left(-\frac{\text{Cos}[k\,r]}{k\,r}-\text{Sin}[k\,r]\right)\text{Sin}[\delta_1]\right)$$

```
sol = Solve[-(Sqrt[k^2 + K^2] r Cos[Sqrt[k^2 + K^2] r] + (-1 + k^2 r^2 + K^2 r^2) Sin[Sqrt[k^2 + K^2] r]) /
    (r (Sqrt[k^2 + K^2] r Cos[Sqrt[k^2 + K^2] r] - Sin[Sqrt[k^2 + K^2] r])) ==
   (cot (Cos[k r]/r + k Sin[k r] - Sin[k r]/(k r^2)) + (-k Cos[k r] + Cos[k r]/(k r^2) + Sin[k r]/r)) /
    (cot (-Cos[k r] + Sin[k r]/(k r)) + (-Cos[k r]/(k r) - Sin[k r])) /. {r -> a}, cot]
```

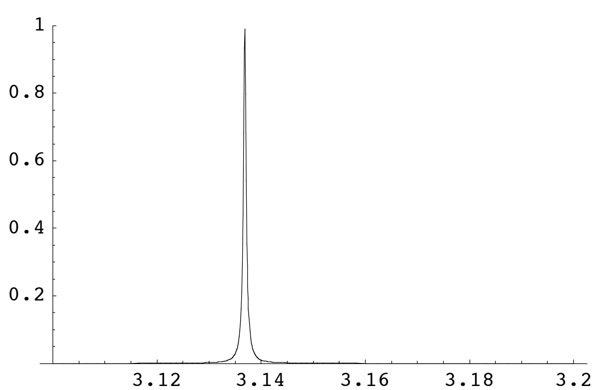
```
{{cot -> (-a k^2 Sqrt[k^2 + K^2] Cos[a k] Cos[a Sqrt[k^2 + K^2]] - K^2 Cos[a k] Sin[a Sqrt[k^2 + K^2]] -
      a k^3 Sin[a k] Sin[a Sqrt[k^2 + K^2]] - a k K^2 Sin[a k] Sin[a Sqrt[k^2 + K^2]]) /
    (-a k^2 Sqrt[k^2 + K^2] Cos[a Sqrt[k^2 + K^2]] Sin[a k] + a k^3 Cos[a k] Sin[a Sqrt[k^2 + K^2]] +
      a k K^2 Cos[a k] Sin[a Sqrt[k^2 + K^2]] - K^2 Sin[a k] Sin[a Sqrt[k^2 + K^2]])}}
```

```
Plot[1/(1 + cot^2) /. sol[[1]] /. {a -> 1} /. {k -> 0.1},
 {K, 0, 10}, PlotRange -> {0, 1}, PlotPoints -> 500]
```



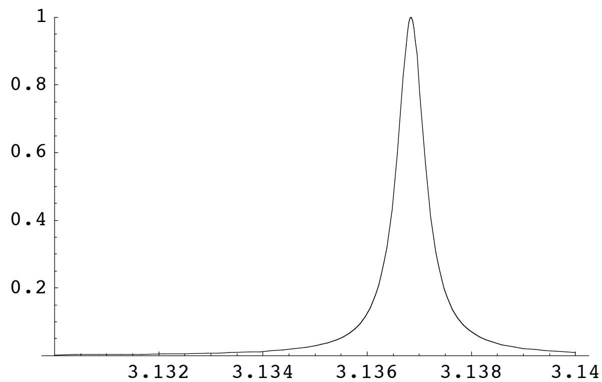
- Graphics -

```
Plot[1/(1 + cot^2) /. sol[[1]] /. {a -> 1} /. {k -> 0.1}, {K, 3.1, 3.2}, PlotRange -> {0, 1}]
```



- Graphics -

```
Plot[1/(1 + cot^2) /. sol[[1]] /. {a → 1} /. {k → 0.1}, {K, 3.13, 3.14}, PlotRange → {0, 1}]
```



- Graphics -

```
firstpeak = FindRoot[cot /. sol[[1]] /. {a → 1} /. {k → 0.1}, {K, 3.137}]
```

{K → 3.13684}

The corresponding value of V_0 at the peak is

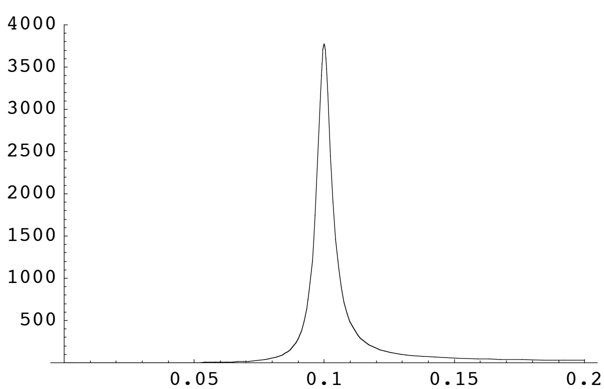
$$\frac{\hbar^2 K^2}{2 m a^2} \text{ /. firstpeak}$$

$$\frac{4.91989\,\hbar^2}{a^2\, m}$$

(b)

The cross section is dominated by the $l = 1$ partial wave around the peak.

```
Plot[(4π/k^2) 3 1/(1 + cot^2) /. sol[[1]] /. {a → 1} /. firstpeak, {k, 0, 0.2}, PlotRange → {0, 4000}]
```



- Graphics -

It shows a prominent peak at $k\,a = 0.1$, and the cross section is much larger than the geometric cross section $4\,\pi\,a^2$.

(c)

We need a precise value for the phase shift at the peak

```
sol[[1]] /. {a → 1, k → 0.1} /. firstpeak
```

```
{cot → -2.81908 × 10^-13}
```

```
δ1 = ArcTan[1/cot /. %]
```

```
-1.5708
```

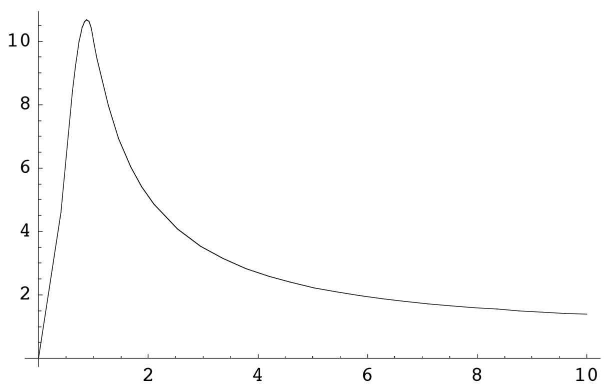
```
Rout /. {δ1 -> -1.5707963267946148`} /. firstpeak /. {a → 1, k → 0.1} /. {r → 1}
```

```
10.0499
```

```
Rin /. {δ1 -> -1.5707963267946148`} /. firstpeak /. {a → 1, k → 0.1} /. {r → 1}
```

```
1.001
```

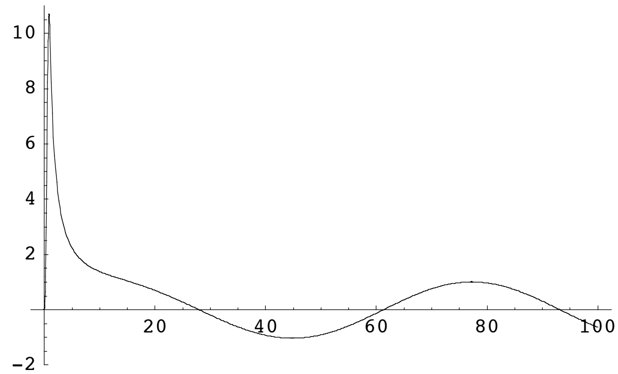
```
Plot[If[r > 1, Rout /. sol[[1]] /. {δ1 -> -1.5707963267946148`}, 10.049875069427088`/1.0010011782275658` Rin] /.
   firstpeak /. {a → 1, k → 0.1}, {r, 0, 10}]
```




```
- Graphics -
```

It appears more-or-less a bound-state solution at small r, because the classical turning point is $\frac{l(l+1)}{r^2} = k^2$ and hence $r = \frac{\sqrt{2}}{k} \simeq 14$ in our case. However, it does oscillate far out in radius as expected for a non-bound state beyond the classical turning point,

```
Plot[If[r > 1, Rout /. sol[[1]] /. {δ1 -> -1.5707963267946148`}, (10.049875069427088` / 1.0010011782275658`) Rin] /.
    firstpeak /. {a → 1, k → 0.1}, {r, 0, 100}, PlotRange → {-2, 11}, PlotPoints → 500]
```



- Graphics -