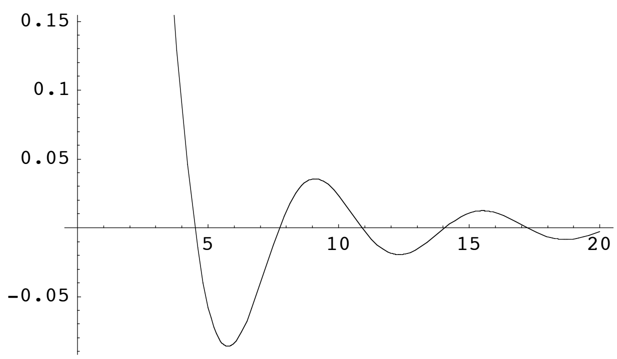
HW #3

1. nuclear size

(a)

As shown in the lecture notes Scattering Theory II, the form factor is
$F(q) = 3\,\frac{\sin q\,a - q\,a\cos q\,a}{(q\,a)^3}$.
The zeros of this function can be found numerically by

```
Plot[3 (Sin[x] - x Cos[x])/x^3, {x, 0, 20}]
```




```
- Graphics -
```

```
FindRoot[3 (Sin[x] - x Cos[x])/x^3, {x, 5}]
```

{x → 4.49341}

```
FindRoot[3 (Sin[x] - x Cos[x])/x^3, {x, 8}]
```

{x → 7.72525}

```
FindRoot[3 (Sin[x] - x Cos[x])/x^3, {x, 11}]
```

{x → 10.9041}

```
FindRoot[3 (Sin[x] - x Cos[x])/x^3, {x, 14}]
```

{x → 14.0662}

(b)

In the case of calcium, we find dips at $\theta = 17$, 30, and 45 degrees, approximately. The experiment uses the electron of energy 750 MeV. Because it is much larger than the rest energy 0.511 MeV, it is relativistic, and the momentum is 750 MeV/c. The scattering angles are related to the momentum transfer by $\hbar^2 q^2 = 4 p^2 \sin^2 \frac{\theta}{2}$. The values of $\hbar q$ are (in MeV/c):

```
N[2 * 750 * Sin[θ/2] /. {θ → 17 * π/180}]
```

```
221.714
```

```
N[2 * 750 * Sin[θ/2] /. {θ → 30 * π/180}]
```

```
388.229
```

```
N[2 * 750 * Sin[θ/2] /. {θ → 45 * π/180}]
```

```
574.025
```

Assuming they correspond to the first three roots, I find in fm (using $\hbar c = 197$ MeV fm)

```
(197 * 4.49) / 221.7
```

```
3.98951
```

```
(197 * 7.73) / 388.2
```

```
3.92275
```

```
(197 * 10.90) / 574.0
```

```
3.74094
```

They are quite consistent with each other. Given that the third dip appears much more uncertain than the first two, I would say the size of the nucleus is about 4.0 fm.

(c)

In the case of lead, there appear to be dips at $\theta = 40$, and 77 degrees, approximately, with 175 MeV electrons. In the data with 250 MeV, we find dips at $\theta = 50$, and 76 degrees, approximately. Following the same steps as in (b), The values of $\hbar q$ are (in MeV/c):

```
N[2 * 175 * Sin[θ/2] /. {θ → 40 * π/180}]
```

```
119.707
```

```
N[2*175*Sin[θ/2] /. {θ → 77*π/180}]
```

217.88

```
N[2*250*Sin[θ/2] /. {θ → 50*π/180}]
```

211.309

```
N[2*250*Sin[θ/2] /. {θ → 76*π/180}]
```

307.831

It appears reasonable to conclude that we have three dips, where the dip at 77 degrees with 175 MeV has the same momentum transfer as the dip at 50 degrees at 250 MeV.

Assuming they correspond to the first three roots, I find in fm (using $\hbar c = 197$ MeV fm)

$$\frac{197*4.49}{119.7}$$

7.38956

$$\frac{197*7.73}{217.9}$$

6.98857

$$\frac{197*10.90}{307.8}$$

6.97628

They are quite consistent with each other. I would say the size of the nucleus is about 7.0 fm.

(d)

Comparing ^{40}Ca and ^{208}Pb, the ratio of the radii is

$$\frac{7.0}{4.0}$$

1.75

while the constant density approximation says it is the cubic root of the ratio of the mass numbers,

$$N\left[\left(\frac{206}{40}\right)^{1/3}\right]$$

1.72691

and they are in a reasonably good agreement. Namely, the density of protons and neutrons in the nuclei appears to have a constant density.

2. Yukawa potential

(a)

The Born approximation gives $f(\theta) = -\frac{2m}{\hbar^2}\ \frac{1}{q} \int_0^\infty r\, V(r) \sin q\, r\, d\, r$.

```
f = - (2 m/ħ^2) (1/q) Integrate[r V0 E^(-r/a)/r Sin[q r], {r, 0, ∞}, Assumptions → {q ∈ Reals && Re[a] > 0}]
```

$$-\frac{2\,\mathrm{m}\,\mathrm{V}_0}{\left(\frac{1}{\mathrm{a}^2}+\mathrm{q}^2\right)\hbar^2}$$

The total cross section is (using $q^2 = 2\,k^2(1-\cos\theta)$),

```
f^2 /. {q^2 → 2 k^2 (1 - c)}
```

$$\frac{4\,\mathrm{m}^2\,\mathrm{V}_0^2}{\left(\frac{1}{\mathrm{a}^2}+2\,(1-\mathrm{c})\,\mathrm{k}^2\right)^2\hbar^4}$$

```
Integrate[2 π %, {c, -1, 1}, Assumptions → a ∈ Reals && k ∈ Reals && a k ≠ 0]
```

$$\frac{16\,\mathrm{a}^4\,\mathrm{m}^2\,\pi\,\mathrm{V}_0^2}{(1+4\,\mathrm{a}^2\,\mathrm{k}^2)\,\hbar^4}$$

This is the total cross section.

(b)

The necessary integral is

$\int \frac{e^{i\,k\,r}}{4\,\pi\,r}\, V(r)\, e^{i\,k\,z}\, d^3\, x = \int \frac{e^{i\,k\,r}}{4\,\pi\,r}\, V_0\, \frac{e^{-r/a}}{r}\, e^{i\,k\,r\,c}\, 2\,\pi\, r^2\, d\, r\, d\, c$

$= \frac{1}{2}\, V_0 \int e^{i\,k\,r}\, e^{-r/a}\, e^{i\,k\,r\,c}\, d\, r\, d\, c = \frac{1}{2}\, V_0 \int e^{i\,k\,r}\, e^{-r/a}\, \frac{e^{i\,k\,r} - e^{-i\,k\,r}}{i\,k\,r}\, d\, r$

$= V_0 \int e^{i\,k\,r}\, e^{-r/a}\, \frac{\sin k\, r}{k\, r}\, d\, r$

I'm lazy and let *Mathematica* do the last integral,

```
Integrate[E^(I k r) E^(-r/a) Sin[k r]/(k r), {r, 0, ∞}, Assumptions -> k ∈ Reals && Re[a] > 0]
```

$$\frac{i\,\mathrm{ArcTanh}\left[\frac{\mathrm{a\,Abs[k]}}{i+\mathrm{a\,k}}\right]\,\mathrm{Sign[k]}}{\mathrm{k}}$$

Therefore we require $\frac{2\,m}{\hbar^2}\ \left|\ \frac{V_0}{k}\ \tanh^{-1}\left(\frac{k\,a}{i+k\,a}\right)\right| \ll 1$.

(c)

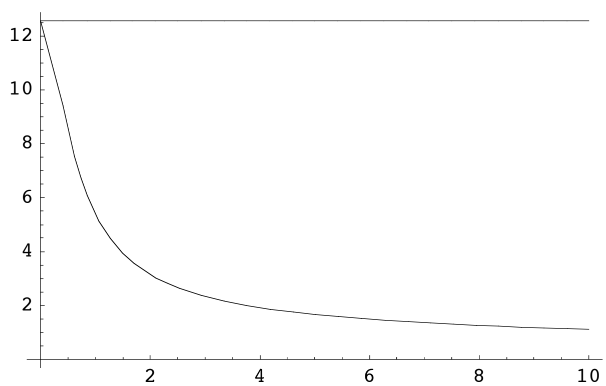
When $k a \ll 1$, the validity requirement reduces to $\frac{2 m}{\hbar^2} \mid V_0 \, a \mid \ll 1$ or $\left| V_0 \right| \ll \frac{\hbar^2}{2 m a}$, which in turn requires the Born cross section to be $\sigma \cong \frac{16 \pi V_0{}^2 m^2 a^4}{\hbar^4} \ll 4 \pi a^2$, hence smaller than the geometric cross section.

When $k a \gg 1$, the inverse hyerbolic tangent is basically logarithm and gives only O(1) enhancement. Therefore the validity requirement reduces to $\frac{2 m}{\hbar^2} \mid \frac{V_0}{k} \mid \ll 1$ or $\left| V_0 \right| \ll \frac{\hbar^2 k}{2 m}$, which in turn requires the Born cross section to be $\sigma \cong \frac{4 \pi V_0{}^2 m^2 a^2}{\hbar^4 k^2} \ll \pi a^2$ again smaller than the geometric cross section.

For intermeate $k a$, we require $\frac{2 m}{\hbar^2} \mid \frac{V_0}{k} \tanh^{-1}(\frac{k a}{i + k a}) \mid \ll 1$, and hence $V_0 \ll \frac{1}{a} \left(\frac{2 m}{\hbar^2} \mid \frac{1}{k a} \tanh^{-1}(\frac{k a}{i + k a}) \mid\right)^{-1}$, The Born cross section then is bounded by $\sigma = \frac{16 a^4 m^2 \pi V_0^2}{(1 + 4 a^2 k^2) \hbar^4} \ll \frac{16 a^2 m^2 \pi}{(1 + 4 a^2 k^2) \hbar^4} \left(\frac{2 m}{\hbar^2} \mid \frac{1}{k a} \tanh^{-1}(\frac{k a}{i + k a}) \mid\right)^{-2}$

Below I plot the bound together with $4 \pi a^2$:

```
In[49]:= Plot[{(16 a^2 m^2 π)/((1 + 4 a^2 k^2) ℏ^4) ((2 m)/ℏ^2 Abs[1/(k a) ArcTanh[(k a)/(I + k a)]])^-2 /. {ℏ → 1, m → 1, a → 1},
    4 π a^2 /. {a → 1}}, {k, 0, 10}]
```



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Out[49]=  - Graphics -
```

Therefore, $\sigma \ll 4 \pi a^2$ within the validity of the Born approximation for *any* k.